# Identification and Estimation of a Search Model with Heterogeneous Consumers and Firms[*†]


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**Abstract**

We propose a model of nonsequential consumer search where consumers and firms differ in search and production costs respectively. We characterize the equilibrium of the game. The search cost distribution is first identified by market shares and prices. The production cost distribution is subsequently identified using a similar strategy to Guerre, Perrigne and Vuong (2000) as the firms' decisions resemble bidders' decisions in a procurement auction. We show the firm's cost density can be estimated at the same convergence rate as the optimal rate in Guerre et al. uniformly over any fixed subset on the interior of the support; it can be made close



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to that rate rate when the subset increases to the full support asymptotically. The difference in rates is due to a pole in the price pdf that we show to be a feature of the equilibrium. We give two extensions of our model with analogous results. One allows for vertically differentiated products. The other has an intermediary. Our simulation study confirms theoretical features of the model. We apply our model to study loan search using UK mortgage data.

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# 1 Introduction

Consumer search cost is one of the classic explanations for why homogeneous goods or services have different prices. Price dispersion can arise in equilibrium for a search model with minimal heterogeneity. An influential paper by Burdett and Judd (1983) showed that a continuous pricing rule can be generated by a mixed strategy Nash equilibrium in a *fixed sample*[1] search model with complete information consisting of infinitely many identical firms and consumers. There, firms are identical because they have the same marginal cost of production and consumers draw search costs from the same distribution. We refer the reader to a survey by Baye, Morgan, and Scholten (2006) for different rationalizations of price dispersions in search and other models.

Hong and Shum (2006) developed an empirical model based on Burdett and Judd (1983). They showed, using just data on prices, nonparametric identification of the firms' marginal costs and parts of the distribution of consumer search costs. Their strategy can also be used to identify an analogous model with finite number of firms (Moraga-González and Wildenbeest (2008)). Identification of the search distribution in these papers is only partial as parts of the support of search cost cannot be identified. Moraga-González, Sándor and Wildenbeest (2013) showed identification on the full support is possible if additional price data from other equilibria are available.

In this paper we propose an empirical model of fixed sample search that allows for heterogeneity across firms as well as consumers. We assume there are a finite number of firms who draw marginal costs from some continuous distribution. Costs are private and firms compete in prices in an *incom-*

[1] In a fixed sample (or nonsequential) search consumers decide from the onset how many price quotes to search for. This stands in contrast to sequential search. The two models are not nested. Morgan and Manning (1985) show the fixed and sequential search models can be optimal in different circumstances. The fixed search model may be more suitable, for example, in applications where time is a factor so that buyers prefer to gather information quickly. Some recent empirical studies found that nonsequential search models provide a better approximation to consumers' search behavior observed in real life (De Los Santos et al. (2012), Honka and Chintagunta (2017)).

*plete information* environment. We analyze both the theoretical and empirical aspects of this model. We make the following contributions:

(i) Provide a system of equations that characterize non-degenerate pure strategy Bayesian-Nash equilibria (BNE) in the model via a fixed-point. This is useful for solving the model particularly for the purpose of counterfactual studies.

(ii) Show both the marginal cost distribution of firms and search cost distribution of consumers can be nonparametrically identified from data on price and market share. Our identification strategy leads to closed-form expressions in terms of the observables that suggest easy to compute estimators.

(iii) Construct a nonparametric estimator for the density of the firm's marginal cost that achieves optimal convergence rate uniformly on any fixed subset in the interior of the support. A slight modification of this estimator can achieve a convergence rate as close to the optimal rate as we like if the subset expands to the whole support. The difference in uniform rates on fixed and expanding support is due to an interesting feature of the equilibrium price density. Our contribution here is a technical one as we show how a two-step density estimator can obtain a near optimal rate when it relies on a preliminary density estimator that estimates an unbounded density.

We provide two motivations for incorporating heterogeneity amongst firms. First, many applications involve only a few firms. It is natural for firms to have different production technologies that are private information in an oligopolistic environment. Another reason is that heterogeneity in firms' is an agnostic way to account for a degree of product differentiation of goods and services that is unobservable (or deemed inconsequential at the time of purchase) to consumers but affects how firms set the price.[2] This conceptually expands the scope of applications of our search model.

While we generalize the model in Moraga-González and Wildenbeest (2008), by allowing heterogeneous firms, due to the information structure, our model is closer to an earlier work by MacMinn (1980). MacMinn assumed firms differ by drawing costs from a uniform distribution. He derived a partial equilibrium result in terms of the optimal pricing rule for firms when all consumers are pre-assigned to search a fixed number of times (cf. Pereira (2005)). Our analysis generalizes his setup because we do not impose any parametric assumption on the marginal cost's distribution and we endogenize consumer search. The decision problem for our firms resembles that of a first-price procurement auction where each bidder has to form an expectation on the number and identity of her competitors. The similarities between search and auction models have been well documented in the theoretical literature, e.g., see McAfee and McMillan (1998) in a mechanism design context; other applications include some job search models from the labor literature.[3]

[2] For example, online sellers of second-hand books or music records have private information about the actual condition of the item that goes beyond the description provided in the offer because they physically own the product.

[3] Well-known labor applications include Postel-Vinay and Robin (2002) and Cahuc, Postel-Vinay and Robin (2006)

As discussed below, identification of our model does not seem possible with price data alone. We exploit the restriction imposed by market shares and price to identify consumers' search distribution. This idea is analogous to linking market shares to choice probabilities, which is the starting point for the identification argument used in the literature on demand for differentiated products (see Berry and Haile (2014)). We show that market shares relate to the equilibrium proportions of consumer search linearly in expectation conditional on price. These proportions can be recovered by solving a linear equation. Following the insight of Hong and Shum (2006), the proportions can be used to identify (finite points of) the search cost distribution. They are also an important ingredient for identifying firm's marginal cost distribution. Our approach on the latter is similar to how Guerre, Perrigne and Vuong (2000, hereafter GPV) identify the distribution of the bidder's latent valuation in a first-price auction model. In particular, we derive the inverse of the equilibrium pricing function explicitly and use it to recover firms' latent costs from observed prices. Our identification strategy is constructive. The parameters of interest can be written explicitly in terms of the joint distribution of observed variables. They can be estimated in closed-form without numerical optimization.

The non-degenerate equilibrium price distribution has an interesting feature. Our analysis reveals its probability density function (pdf) generally has a *pole* at the upper support. I.e., the pdf asymptotes to infinity at that point. Intuitively this happens because there are consumers who search only one time and will pay whatever price the firm charges up to their valuation of the good. Correspondingly, firms have an incentive to charge close to that price. Poles also appear in other structural models such as the equilibrium distributions of bids in a first price auction with binding reserved price (see Section 4 in GPV) and wages in a job search model (Bontemps, Robin and van den Berg (2000)). These authors show their respective estimators of bidder's valuation and firm's productivity densities attain the optimal uniform convergence rates derived in GPV on any fixed interval in the interior of the support. Our density estimator of the firm's costs achieves the same rates. The pole, however, prevents the optimal convergence rates of these estimators to hold over intervals that expand towards it. We apply the strategy in Srisuma (2023), who studied convergence rates of density estimators when there is a pole, to construct an estimator that can achieve any uniform convergent rate that is slower than the optimal benchmark (without a pole) where uniformity is taken over a suitably expanding support that increases to the full support asymptotically.

We provide two extensions of our baseline model where our closed-form identification and convergence results apply. One is when there is vertical product differentiation known to consumers and firms but not to the econometrician. This model can be seen as an incomplete information coun-

who model on-the-job search as a sequential auction over the worker between the current and prospective employer. A job search model that is closer to ours is the work by Bontemps, Robin and van den Berg (1999) as they allow for heterogeneous opportunity costs of keeping jobs among workers and continuous productivity among firms.

terpart to the model in Wildenbeest (2011) that uses quality to explain systematic price differences between firms. The other is a model with an intermediary (i.e., a broker), where consumers with very high search costs purchase through an intermediary who conduct an exhaustive search for a fee. Our latter model is inspired by a recent empirical model proposed by Salz (2020).

There are noteworthy similarities and differences between Salz's model and ours. Salz treats the search and broker markets separately in his model. When there is no intermediary his search model is the same as ours. In the pure search setting, Salz's numerical exercise (Appendix B.1) indicates one cannot identify a search model with heterogeneous consumers and firms using price data alone. With an intermediary present, Salz showed search cost distribution can be identified if the firms' cost distribution is known a priori. It is plausible to assume the knowledge of firms' cost distribution in applications where the same firms in both the search and broker markets[4]. We combine the search and broker market in one framework in anticipation of our application where there are no separate physical markets. Our identification strategy, which uses market share and price, can identify a search model with or without an intermediary.

For the ease of notation and clarity of idea, the paper presents the identification arguments and theoretical results in the setting where there are no observable characteristics, and each firm charges every consumer the same price. The identification results immediately extend conditional on observables. Variables that are specific to buyers or firms generate variation in equilibria that reduces the degree of partial identification as alluded to above (cf. Moraga-González et al. (2013)). Relatedly, in practice, a parametric model may be preferred as nonparametric estimators suffer from the curse of dimensionality with many conditioning variables. Parametric features can be easily embedded in our identification strategy. Identification and estimation a model where consumers get individualized prices is more challenging. In this case, we propose a parametric-framework where our identifying assumption and estimation strategy developed for the posted-price case can be used. Our approach can also be applied to models with vertically differentiated products or an intermediary.

Our numerical studies consist of a Monte Carlo simulation and an empirical application. The simulation study confirms the pole exists and shows nonparametric estimators that ignore the pole can perform poorly in its vicinity. Our application uses mortgage data from the UK to estimate a model with mortgage brokers playing the role of intermediaries. The latter involves a rich dataset and we estimate it parametrically.

We organize the rest of the paper as follows. Section 2 presents the model and characterizes the equilibrium of the game. We give identification results in Sections 3 and show how they lead to estimators with desirable properties in Section 4. Section 5 studies two extensions of our baseline

[4]This is the case in his application where auctions took place in the broker market. Salz had the auction data and could identify the cost distribution directly using GPV.

model. Section 6 and 7 contain a Monte Carlo exercise and empirical application respectively. Section 8 concludes with some discussions. The proofs of all results not given in the main text can be found in the Appendix.

# 2 Model

Consider a model in which there is a unit mass of consumers and a finite number of firms. Each consumer has an inelastic demand for a single unit of a good supplied by the firms. Consumers differ by search costs. They have a belief on the price distribution and employ a nonsequential search strategy to decide on the number of firms to visit and purchase at the lowest price. Firms differ by production costs. They form beliefs about consumer search behavior and competing firms' pricing strategies, and set their price to maximize expected profits.

The primitives of our search model are $G(\cdot)$ and $H(\cdot)$ that respectively represent the search cost cdf and production cost cdf. The number of firms, denoted by $I$, is finite and known. We model consumers in the same way as Moraga-González and Wildenbeest (2008), Moraga-González, Sándor and Wildenbeest (2013), and Sanches, Silva and Srisuma (2018). These only differ from Hong and Shum (2006) in that the latter assumed $I$ is infinite. We describe the decision problem and the best response for the consumers in Section 2.1. The aforementioned papers assume firms have identical production costs and that is common knowledge. We assume costs differ across firms and they are private information. We describe the firms' decision problem and derive their best response in Section 2.2. We define the equilibrium of our game in Section 2.3.

## 2.1 Consumers

All consumers have the same valuation of the object at some finite and positive $\overline{P}$. Each consumer draws a search cost $c$, which is assumed to be a continuous random variable support on $\left[\underline{C}, \overline{C}\right] \subset \mathbb{R}^{+}$ with cdf $G(\cdot)$. A consumer with search cost $c$ faces the following decision problem when a purchase is always made:

$$\min_{1 \leq k \leq I} \; ck + \mathbb{E}_F\left[P_{(1:k)}\right].$$

We use $P_{(k:k')}$ to denote the $k-$th order statistic from $k'$ i.i.d. random variables of prices with some arbitrary distribution; $P_{(1:k)}$ denotes the minimum of such $k$ prices. The game is symmetric as all firms have equal probability of being found. We use $\mathbb{E}_F[\cdot]$ to denote an expectation where the random prices have distribution described by the cdf $F(\cdot)$.

**Consumer's Best Response**

The marginal saving from searching one more firm after having searched $k$ firms is:

$$\Delta_k(F) \equiv \mathbb{E}_F\left[P_{(1:k)}\right] - \mathbb{E}_F\left[P_{(1:k+1)}\right]. \tag{1}$$

$\Delta_k(F)$ is non-increasing in $k$ because $\mathbb{E}_F\left[P_{(1:k)}\right]$ is non-increasing in $k$. When price has a differentiable cdf, $\mathbb{E}_F\left[P_{(1:k)}\right]$ is strictly increasing and

$$\Delta_k(F) = \int F(p)(1-F(p))^k \, dp. \tag{2}$$

The optimal behavior for a consumer that draws $c > \Delta_1(F)$ is to search once and search $k > 1$ if $c \in [\Delta_k(F), \Delta_{k-1}(F))$. For the purpose of defining equilibrium (see below), we can state the best response for consumers in terms of proportions of consumer search. In what follows, we use $\mathcal{F}$ to denote a set of all price cdfs and $\mathbb{S}^{I-1}$ to denote a unit simplex in $\mathbb{R}^{I+}$.

LEMMA 1. *A consumer's best response is a map $\sigma_D : \mathcal{F} \to \mathbb{S}^{I-1}$ such that for any $F$ in $\mathcal{F}$,*

$$\sigma_D(F) = \begin{cases} 1 - G(\Delta_k(F)) & \text{for } k = 1 \\ G(\Delta_{k-1}(F)) - G(\Delta_k(F)) & \text{for } 1 < k < I \\ G(\Delta_{I-1}(F)) & \text{for } k = I \end{cases}. \tag{3}$$

*where $\{\Delta_k\}_{k=1}^{I-1}$ is defined in (1).*

Note that equation (3) holds irrespective whether the first price is free or not. The key feature of the best response in Lemma 1 is we can sort consumers so that those drawing higher costs cannot search more than those with lower costs. Such structure is accommodated by non-linear cost functions that allow some economy or dis-economy of scale if one has a prior knowledge to impose them.[5]

[5] For example, suppose $\Delta\chi(c,k) = \chi_1(c)\chi_2(k)$ where $\chi_1$ is strictly increasing and $\chi_2$ is positive. This includes the linear cost as a special case when $\chi_1(c) = c$ and $\chi_2(k) = 1$. If $\{\Delta_k/\chi_2(k)\}_{k=1}^{I-1}$ is strictly decreasing, then equation (3) in Lemma 1 can be generalized to:

$$\sigma_D(F) = \begin{cases} 1 - G\left(\chi_1^{-1}(\Delta_k(F)/\chi_2(k))\right) & \text{for } k = 1 \\ G\left(\chi_1^{-1}(\Delta_{k-1}(F)/\chi_2(k-1))\right) - G\left(\chi_1^{-1}(\Delta_k(F)/\chi_2(k))\right) & \text{for } k > 1 \end{cases},$$

and the proof strategy for Proposition 1 remains applicable. To see how $\Delta\chi$ accommodates both economy and dis-economy of scale on search: (i) suppose $\chi_2$ is increasing (dis-economy of scale case) then $\{\Delta_k/\chi_2(k)\}_{k=1}^{I-1}$ is strictly decreasing for any $\Delta_k$ that comes from a non-degenerate price distribution; (ii) if $\chi_2$ is decreasing (economy of scale case) then $\{\Delta_k/\chi_2(k)\}_{k=1}^{I-1}$ can be strictly decreasing for some price distribution that needs to be determined on a case-by-case basis.

## 2.2 Firms

Firm $i$ draws a marginal cost of production $R_i$. $R_i$ is assumed to be a continuous random variable supported on $\left[\underline{R},\overline{R}\right]\subset\mathbb{R}^+$ with cdf $H(\cdot)$ where $\overline{R}$ is finite. Firm costs are private information that are independent from each other. Under symmetry, firm $i$ then faces the following decision problem:

$$\max_p \Lambda\left(p,R_i;\mathbf{q}\right), \text{ where}$$
$$\Lambda\left(p,R_i;\mathbf{q}\right) = \left(p-R_i\right)\sum_{k=1}^{I} q_k\frac{k}{I}\mathbb{P}\left[P_{(1:k-1)}>p\right].$$

Here $\mathbf{q}=\left(q_1,\ldots,q_I\right)^\top$ denotes a vector in $\mathbb{S}^{I-1}$, where $q_k$ denotes the proportion of consumers searching $k$ firms. The term $\frac{k}{I}$ is the probability that firm $i$ gets included when $k$ firms are sampled. Note that when $q_I=1$, our firm's decision problem is the same as the bidder's problem in a standard first-price procurement auction.

### Firm's Best Response

We consider a pricing strategy $\beta:\left[\underline{R},\overline{R}\right]\to\left[\underline{P},\overline{P}\right]\subset\mathbb{R}$ that is strictly increasing almost everywhere and satisfies $\beta\left(\overline{R}\right)=\overline{R}$. The latter is the zero profit condition. We assume $\overline{R}=\overline{P}$, so that firms always produce and a purchase is always made. For any $\mathbf{q}\in\mathbb{S}^{I-1}$, we can define $\Lambda^*(\cdot;\mathbf{q})$ to be the value function for a representative firm when all players are assumed to employ a strictly increasing optimal pricing strategy that we denote by $\beta(\cdot;\mathbf{q})$. We denote its inverse, $\beta^{-1}(\cdot;\mathbf{q})$ by $\xi(\cdot;\mathbf{q})$.

$$\Lambda^*\left(r;\mathbf{q}\right)=\left(\beta\left(r;\mathbf{q}\right)-r\right)\sum_{k=1}^{I} q_k\frac{k}{I}\left(1-H\left(\xi\left(\beta\left(r;\mathbf{q}\right);\mathbf{q}\right)\right)\right)^{k-1}.$$

Then, by the envelope theorem (Milgrom and Segal (2002)),

$$\left.\frac{d}{dr}\Lambda^*\left(r;\mathbf{q}\right)\right|_{r=R} = -\sum_{k=1}^{I} q_k\frac{k}{I}\left(1-H\left(R\right)\right)^{k-1}, \text{ and}$$
$$\Lambda^*\left(\overline{R};\mathbf{q}\right)-\Lambda^*\left(R;\mathbf{q}\right) = -\sum_{k=1}^{I} q_k\frac{k}{I}\int_{s=R}^{\overline{R}}\left(1-H\left(s\right)\right)^{k-1}ds.$$

Solving this gives the solution of the firm's maximization problem, where for all $r$:

$$\beta\left(r;\mathbf{q}\right)=r+\frac{\sum_{k=1}^{I} q_k k\int_{s=r}^{\overline{R}}\left(1-H\left(s\right)\right)^{k-1}ds}{\sum_{k=1}^{I} q_k k\left(1-H\left(r\right)\right)^{k-1}}. \tag{4}$$

It can be verified that $\beta(\cdot;\mathbf{q})$ is continuous and non-decreasing on $\left[\underline{R},\overline{R}\right]$ as well as satisfying $\beta\left(\overline{R}\right)=\overline{R}$. Furthermore, suppose $H(\cdot)$ is differentiable with a positive pdf, $h(\cdot)$. Differentiating the expression above gives,

$$\beta'(r;\mathbf{q}) = \frac{h(r)\left(\sum_{k=2}^{I} q_k k(k-1)(1-H(r))^{k-2}\right)\left(\sum_{k=1}^{I} q_k k \int_{s=r}^{\overline{R}} (1-H(s))^{k-1} ds\right)}{\left(\sum_{k=1}^{I} q_k k(1-H(r))^{k-1}\right)^2}. \tag{5}$$

This shows $\beta(\cdot;\mathbf{q})$ is strictly increasing on $\left[\underline{R}, \overline{R}\right]$ whenever $q_1 < 1$ and $\beta(r;\mathbf{q}) = \overline{R}$ for all $r$ when $q_1 = 1$.

We define the firm's best response to the consumers in terms of the distribution of $\beta(R_i;\mathbf{q})$.

LEMMA 2. *The firm's best response is a map* $\sigma_S : \mathbb{S}^{I-1} \to \mathcal{F}$ *such that for any* $\mathbf{q}$ *in* $\mathbb{S}^{I-1}$, $\sigma_S(\mathbf{q})$ is the cdf of $\beta(R_i;\mathbf{q})$ *where* $\beta(\cdot;\mathbf{q})$ *is defined as in (4).*

## 2.3 Equilibrium

We define a symmetric equilibrium for our game by any pair of consumer search proportions and induced cdf for firm's pricing strategy that simultaneously satisfy the best responses on both the demand and supply side.

DEFINITION 1. *A pair* $(\mathbf{q}, F) \in \mathbb{S}^{I-1} \times \mathcal{F}$ *is a symmetric equilibrium if* $\mathbf{q} = \sigma_D(F)$ *and* $F = \sigma_S(\mathbf{q})$, *where* $\sigma_S(\cdot)$ *and* $\sigma_D(\cdot)$ *are defined in Lemmas 1 and 2 respectively.*

An equilibrium with degenerate price distribution always exists in our model. This occurs when all consumers search once and all firms set the monopoly price, i.e. $\beta_M(r;\mathbf{q}_M) = \overline{R}$ for all $r$. I.e., $\mathbf{q}_M$ is such that $q_{1M} = 1$ and $q_{kM} = 0$ for $k \neq 1$ (cf. Diamond (1971)). Such equilibrium is not suitable in many applications where prices differ. We focus on the case when $\beta(\cdot;\mathbf{q})$ is strictly increasing. Theorem 1 characterizes such equilibria by $\mathbf{q}$ that satisfies (3) and (4) simultaneously.

THEOREM 1. *In a symmetric equilibrium* $(\mathbf{q}, F) \in \mathbb{S}^{I-1} \times \mathcal{F}$, *where the equilibrium pricing strategy is strictly increasing,* $\mathbf{q}$ *satisfies the following system of equations:*

$$q_k = \begin{cases} 1 - G\left(\int F(p)(1-F(p))\,dp\right) & \text{for } k = 1 \\ G\left(\int F(p)(1-F(p))^{k-1} dp\right) - G\left(\int F(p)(1-F(p))^{k} dp\right) & \text{for } 1 < k < I \end{cases}, \tag{6}$$

*where* $F(p) = H(\xi(p;\mathbf{q}))$ *for all* $p \in \left[\underline{P}, \overline{P}\right]$.

The characterization above shows that an equilibrium can be summarized by a fixed-point, which is useful for solving the model. In general there may more multiple equilibria. We are not aware of a uniqueness result in this context.

In subsequent sections we consider the econometric problem of identifying and estimating the model primitives from data generated from a particular equilibrium. We will henceforth drop the indexing arguments of equilibrium objects that are made explicit in this section for the purpose of defining best response and equilibrium. E.g. $\beta(\cdot;\mathbf{q})$ becomes $\beta(\cdot)$, $\mathbb{E}_F[\cdot]$ becomes $\mathbb{E}[\cdot]$ etc.

# 3 Nonparametric Identification

We assume to observe $\{(Y_{im}, P_{im})\}_{i=1,m=1}^{I,M}$ where $Y_{im}$ is *observed* market share and $P_{im}$ is price of firm $i$ in market $m$ such that the data is generated from a single equilibrium. We distinguish $Y_{im}$ from the *theoretical* market share. We formally define both below. Here $M$ is the total number of markets and we will use a large $M$ asymptotics framework.

ASSUMPTION D. *$\{(Y_{im}, P_{im})\}_{i=1,m=1}^{I,M}$ is a sequence of random variables such that:*

*(i) there exists $(\mathbf{q}, F) \in \mathbb{S}^{I-1} \times \mathcal{F}$ with $q_1 \in (0,1)$ so that $P_{im} = \beta(R_{im}) \equiv \beta(R_{im};\mathbf{q})$ where $\beta(\cdot;\mathbf{q})$ has been defined in (4) and $\{R_{im}\}_{i=1,m=1}^{I,M}$ is i.i.d. with a continuous density that is positive and finite almost everywhere on $[\underline{R}, \overline{R}]$;*

*(ii) $\{(Y_{1m}, \ldots, Y_{Im})\}_{m=1}^{M}$ is i.i.d. such that the joint distribution of $(Y_{im}, P_{im})$ satisfies,*

$$\mathbb{E}[Y_{im}|P_{im}] = \sum_{k=1}^{I} q_k \frac{k}{I}(1 - F(P_{im}))^{k-1}. \tag{7}$$

Assumption D(i) assumes observed prices are a random sample from an equilibrium of a search model. $q_1 < 1$ ensures $\beta(\cdot;\mathbf{q})$ is strictly increasing and price has a continuous distribution. Having $q_1 = 0$ does not affect our identification strategy and it simplifies our asymptotic analysis. However, we expect $q_1 > 0$ to be the norm in many applications and it has an econometric implication (due to the pole in the price density (see Lemma 4(b))). We assume the more difficult case is on hand rather than having to treat two separate cases. Equation (7) in Assumption D(ii) states a defining property of market shares as the RHS of the equation has the interpretation of the ex-ante probability of firm $i$ winning a sale by setting price to be $P_{im}$. Assumption D(ii) also assumes shares across markets are i.i.d., but it allows shares to be correlated within a market.

Observed and theoretical market shares generally differ. The former is an aggregation of decisions from a finite number of consumers and the latter aggregates decisions from a continuum of consumers. To define these shares formally, omitting the market index, we denote the search cost for consumer $b$ by $c_b$ and let $C_k := [\Delta_k, \Delta_{k-1}]$ for $k > 0$ where $\{\Delta_k\}_{k=0}^{I-1}$ denote the search costs where consumers are indifferent in making $k$ and $k+1$ searches. Let $D_{bi}$, $\ell_{bi}$, and $\ell_{bi\mathcal{A}}$ be binary variables that takes value 1 if consumer respectively $b$ purchases from firm $i$, searches only at firm $i$, and searches at firm $i$ along with $k-1$ other firms in the set $\mathcal{A} \in \mathcal{I}_k^i := \left\{ \mathcal{A} = \bigcup_{j \in \mathcal{I} \setminus \{i\}} \{j\} \middle| |\mathcal{A}| = k-1 \right\} \subseteq \mathcal{I} := \{1, \ldots, I\}$. The micro-foundation for market share of firm $i$ is based on an individual's purchasing decision,

$$D_{bi} = \ell_{bi} \mathbf{1} \left[ c_b > \Delta_1 \right] + \sum_{k=2}^{I} \sum_{\mathcal{A} \in \mathcal{I}_k^i} \ell_{bi\mathcal{A}} \mathbf{1} \left[ c_b \in C_k, P_i < \min_{j \in \mathcal{A}} \{P_j\} \right]. \tag{8}$$

When each firm has an equal chance of being found, it is easy to verify that $\mathbb{E}\left[D_{bi}|P_i\right]$ leads to the expression on the RHS of (7). When $Y_{im}$ is defined as an average of $D_{bi}$ over $B$ i.i.d. consumers, (7) follows. As $B \to \infty$, by the law of large numbers, $Y_{im}$ converges to the theoretical market share:

$$\overline{Y}_{im} := \frac{q_1}{\mathcal{C}_1^I} + \sum_{k=2}^{I} q_k \frac{\sum_{\mathcal{A} \in \mathcal{I}_k^i} \mathbf{1}\left[P_{im} < \min_{j \in \mathcal{A}} \{P_{jm}\}\right]}{\mathcal{C}_k^I},$$

where $\mathcal{C}_k^I := \frac{I!}{(I-k)!k!}$. Note that $\mathbb{E}\left[\overline{Y}_{im}|P_{im}\right] = \mathbb{E}\left[Y_{im}|P_{im}\right]$, because $\sum_{\mathcal{A} \in \mathcal{I}_{ik}} \mathbb{E}\left[P_{im} < \min_{j \in \mathcal{A}} \{P_{jm}\} | P_{im}\right] = \mathcal{C}_{k-1}^{I-1} \left(1 - F\left(P_{im}\right)\right)^{k-1}$ and $\mathcal{C}_{k-1}^{I-1} / \mathcal{C}_k^I = \frac{k}{I}$. The discrepancy between $Y_{im}$ and $\overline{Y}_{im}$ thus represents an approximation error of the model. The error, $\epsilon_{im} := Y_{im} - \overline{Y}_{im}$, exhibits a property of a classical measurement error since $\mathbb{E}\left[\epsilon_{im}|P_{im}\right] = 0$. Note that $\left(\overline{Y}_{im}, \overline{Y}_{jm}\right)$ are correlated as they are jointly determined by prices and we expect $\left(Y_{im}, Y_{jm}\right)$ to be correlated as a consequence.

In Section 3.1 we consider identification on the demand side. We first identify $\mathbf{q}$ using (7), based on $\{(Y_{im}, P_{im})\}_{i=1,m=1}^{I,M}$, which can then be used to identify $G(\cdot)$. We identify $H(\cdot)$ in Section 3.2. For the latter, it suffices to show how to recover firm costs, $\{R_{im}\}_{i=1,m=1}^{I,M}$. In both Sections 3.1 and 3.2 we take $F(\cdot)$ and the joint distribution of $(Y_{im}, P_{im})$ for any $(i,m)$ to be known. Both of these objects are nonparametrically identified under Assumption D when $M \to \infty$.

## 3.1 Consumers

Let $X_{im}$ be a vector in $\mathbb{R}^I$ such that $\left(X_{im}\right)_k = \frac{k}{I}\left(1 - F\left(P_{im}\right)\right)^{k-1}$. We can write (7) as

$$Y_{im} = X_{im}^{\top} \mathbf{q} + \varepsilon_{im}, \tag{9}$$

where $\varepsilon_{im}$ satisfies $\mathbb{E}\left[\varepsilon_{im}|P_{im}\right] = 0$. We can then identify $\mathbf{q}$ as the solution of a least squares problem.

Lemma 3. *Suppose Assumption D holds. If* $\mathbb{E}\left[X_{im}X_{im}^{\top}\right]$ *has full rank then* $\mathbf{q}$ *is identified.*

We now treat both $\mathbf{q}$ and $F(\cdot)$ as known and use them to identify $G(\cdot)$ at $\{\Delta_k\}_{k=1}^{I-1}$

Proposition 1. *Suppose Assumption D holds. Then* $G(\Delta_k)$ *is identified for* $k = 1, \ldots, I-1$.

Proof. From (4), we see that $G(\Delta_k) = 1 - \sum_{k'=1}^{k} q_{k'}$ for $k = 1, \ldots, I-1$. The proof follows since both $\{\Delta_k\}_{k=1}^{I-1}$ and $\mathbf{q}$ are identified. In particular, note that $\Delta_k$ is a functional of $F(\cdot)$ for all $k$ (see (1) and (2)) and $\mathbf{q}$ is identified from Lemma 3. ■

Our identification strategy for $\mathbf{q}$ is different to the method used in a complete information model (Hong and Shum (2006), Moraga-González and Wildenbeest (2008)). Once $\mathbf{q}$ is identified, however, each $G(\Delta_k)$ can be identified in the same way.

Like prior results, $G(\cdot)$ is partially identified as we can identify it at $\{\Delta_k\}_{k=1}^{I-1}$. The degree of non-identification can be reduced if there is exogenous variation across markets. For example, suppose there are $L$ market types where consumers draw search costs from the same distribution but firms production costs have different distribution across types and/or the number of firms may vary with $L$. Moraga-González, Sándor and Wildenbeest (2013) propose this identification strategy in the complete information context. The same idea applies to our setting. We illustrate in Section 7 how to find and exploit such variation empirically.

## 3.2 Firms

We can identify $H(\cdot)$ by inverting latent production costs from observed prices. Lemma 4 provides key properties of the price density and gives an explicit formula for the inverse of the pricing strategy in terms of $\mathbf{q}$ and the price distribution.

Lemma 4. *Suppose Assumption D(i) holds. Then:*

*(a)* $\inf_{p\in[\underline{P},\overline{P}]} f(p) > 0$*;*

*(b)* $\lim_{p\to\overline{P}} f(p) = \infty$*;*

*(c) the inverse of the equilibrium pricing strategy,* $\xi : [\underline{P},\overline{P}] \to [\underline{R},\overline{R}]$*, takes the following form*

$$\xi(p) = p - \frac{\sum_{k=1}^{I} q_k k (1-F(p))^{k-1}}{f(p)\sum_{k=2}^{I} q_k k (k-1)(1-F(p))^{k-2}}, \tag{10}$$

*and* $\xi(\overline{P}) = \overline{R}$*.*

When $q_1 = 0$, it can be shown that $f(\cdot)$ is bounded away from zero on $[\underline{P},\overline{P})$ and $f(\overline{P})$ does not need to be infinite for $\xi(\overline{P}) = \overline{R}$.

Proposition 2. *Suppose Assumption D holds. Then* $H(\cdot)$ *is identified.*

Proof. Under Assumption D, $\xi(\cdot)$ is identified. Therefore we can recover $R_{im}$ from $\xi(P_{im})$ for all $i, m$. ■

# 4 Estimation and Convergence Rates

We now look to estimate $h(\cdot)$ at the best possible convergence rate. We focus on this problem because $h(\cdot)$ is the most difficult object to estimate in our model in the sense that its estimator has the slowest convergence rate amongst estimators of other identifiable objects. Along the way, we will discuss estimators of other parameters in the model. Particularly, parameters on the demand side can be estimated at the parametric rate and our discussion on them will be brief.

We consider two separate cases. First, we consider the uniform convergence for $h(\cdot)$ over any fixed closed interval that lies in the interior of $\left[\underline{R}, \overline{R}\right]$. In this case we can provide an estimator for $h(\cdot)$ that achieves the same optimal convergence rate as the GPV estimator. Second, we consider uniform convergence over an expanding interval that approaches $\left[\underline{R}, \overline{R}\right]$ as the sample size increases. In this case we will provide another estimator for $h(\cdot)$ that can converge at any slower rate than the one achievable over a fixed support. The reason for a slower convergence rate is the latter accounts for the pole ($\lim_{p \to \overline{P}} f(p) = \infty$). Our estimator for $h(\cdot)$ in both cases will be based on kernel smoothing using estimated $\{R_{im}\}_{i=1,m=1}^{I,M}$, which is to be obtained through the estimated inverse of the pricing function (see (10)). In particular, the inverse of the pricing function depends on $(\mathbf{q}, F(\cdot), f(\cdot))$ that have to be estimated.

To study convergence rates, we have to specify the degree of smoothness of $H(\cdot)$.

Assumption R. *$H(\cdot)$ admits upto $\tau + 1$ continuous derivatives on $\left[\underline{R}, \overline{R}\right]$ for some $\tau \geq 1$.*

Lemma 5. *Suppose Assumptions D and R hold. Then $f(\cdot)$ admits upto $\tau + 1$ continuous derivatives on $\left[\underline{P}, \overline{P}\right)$ for the same $\tau$ as in Assumption R.*

Lemma 5 says that $f(\cdot)$ has the same degree of smoothness as $H(\cdot)$ everywhere other than at $\overline{P}$. We next define estimators for $(\mathbf{q}, f(\cdot), F(\cdot))$ and discuss their convergence rates under Assumptions D and R.

**An estimator for $F(\cdot)$**

A natural estimator for $F(\cdot)$ is the empirical cdf, defined as

$$\widehat{F}(p) = \frac{1}{MI} \sum_{m=1}^{M} \sum_{i=1}^{I} \mathbf{1}\left[P_{im} \leq p\right] \quad \text{for all } p. \tag{11}$$

It is well-known from Donsker's theorem that $\sqrt{M}\left(\widehat{F}(\cdot) - F(\cdot)\right)$ converges weakly to a Gaussian process on $\left[\underline{P}, \overline{P}\right]$. Then by the continuous mapping theorem, $\sup_{p \in \left[\underline{P}, \overline{P}\right]} \left|\widehat{F}(p) - F(p)\right| = O_p\left(1/\sqrt{M}\right)$.

**An estimator for q**

We suggest to estimate $\mathbf{q}$ by least squares. Let $\mathbf{Y}_m = (Y_{1m}, \ldots, Y_{Im})^\top$, $\mathbf{e}_m = (\varepsilon_{1m}, \ldots, \varepsilon_{Im})^\top$ and $\mathbf{X}_m$ be an $I \times I$ matrix such that $(\mathbf{X}_m)_{ik} = \frac{k}{I}(1 - F(P_{im}))^{k-1}$. Vectorize $\mathbf{Y}_m$, $\mathbf{X}_m$ and $\mathbf{e}_m$ across

$m$ to form $\mathbf{Y} = \left[\mathbf{Y}_1^\top : \ \cdots \ : \mathbf{Y}_M^\top\right]^\top$, $\mathbf{X} = \left[\mathbf{X}_1^\top : \cdots : \mathbf{X}_M^\top\right]^\top$ and $\mathbf{e} = \left[\mathbf{e}_1^\top : \ \cdots \ : \mathbf{e}_M^\top\right]^\top$ respectively. Then a vector version of (9) is,

$$\mathbf{Y} = \mathbf{X}\mathbf{q} + \mathbf{e}.$$

$F(\cdot)$ is unknown and has to be estimated. Let $\widehat{\mathbf{X}}$ be the feasible counterpart of $\mathbf{X}$ where $F(\cdot)$ is replaced by $\widehat{F}(\cdot)$. Then,

$$\begin{aligned} \widehat{\mathbf{q}} &= \left(\widehat{\mathbf{X}}^\top \widehat{\mathbf{X}}\right)^{-1} \widehat{\mathbf{X}}^\top \mathbf{Y}, \qquad (12) \\ &= \mathbf{q} + \mathbf{a}_M + \mathbf{b}_M, \text{ where} \\ \mathbf{a}_M &= \left(\mathbf{X}^\top \mathbf{X}\right)^{-1} \mathbf{X}^\top \mathbf{e}, \\ \mathbf{b}_M &= \left(\left(\widehat{\mathbf{X}}^\top \widehat{\mathbf{X}}\right)^{-1} \widehat{\mathbf{X}}^\top - \left(\mathbf{X}^\top \mathbf{X}\right)^{-1} \mathbf{X}^\top\right) \mathbf{Y}. \end{aligned}$$

Using asymptotic theory for clustered samples (e.g. see Hansen and Lee (2019)), $\|\mathbf{a}_M\| = O_p\left(1/\sqrt{M}\right)$ as $\frac{1}{M}\mathbf{X}^\top\mathbf{X} = \frac{1}{M}\sum_{m=1}^M \left(\sum_{i=1}^I X_{im} X_{im}^\top\right)$ and $\frac{1}{\sqrt{M}}\mathbf{X}^\top\mathbf{e} = \frac{1}{\sqrt{M}}\sum_{m=1}^M \left(\sum_{i=1}^I X_{im}\varepsilon_{im}\right)$ would satisfy a law of large numbers and central limit theorem respectively. Since $\left(\widehat{\mathbf{X}}^\top \widehat{\mathbf{X}}\right)^{-1} \widehat{\mathbf{X}}^\top$ is a smooth functional of $\widehat{F}(\cdot)$, it can also be verified by applications of the continuous mapping theorem that $\|\mathbf{b}_M\| = O_p\left(1/\sqrt{M}\right)$. Thus, $\|\widehat{\mathbf{q}} - \mathbf{q}\| = O_p\left(1/\sqrt{M}\right)$.

Furthermore, since $\Delta_k$ is a functional of $F(\cdot)$, we can estimate $G(\Delta_k)$ using $\widehat{\mathbf{q}}$ and $\widehat{F}(\cdot)$ based on the constructive identification result in Proposition 1. Such estimator will be a smooth functional of $\widehat{F}(\cdot)$ and have a $\sqrt{M}-$convergence rate. Estimating $G(\cdot)$ as a curve is also possible when there are data from different equilibria that can identify more points on the support of the search cost. In this case, Sanches, Silva and Srisuma (2018) proposed a series estimator that pooled data across equilibria based on using estimated $\Delta_k$ and $G(\Delta_k)$ as generated regressor and regressand respectively; they also derive the convergence rate of such estimator. The same type of estimator can also be constructed here. We refer the reader to Section 4 of Sanches, Silva and Srisuma (2018) for further details.

**An estimator for $f(\cdot)$**

Consider the following kernel density estimator for $f(\cdot)$,

$$\widehat{f}(p) = \frac{1}{MIb_{f,M}} \sum_{m=1}^M \sum_{i=1}^I K\left(\frac{P_{im} - p}{b_{f,M}}\right) \quad \text{for all } p, \qquad (13)$$

where $K(\cdot)$ is a $(\tau+1)-$th higher order kernel function and $b_{f,M}$ is a bandwidth that is proportional to the optimal bandwidth that converges to zero at the rate $\left(\frac{\log M}{M}\right)^{\frac{1}{2\tau+3}}$, see Härdle (1991). Let $\eta_M^* \equiv \left(\frac{\log M}{M}\right)^{\frac{\tau+1}{2\tau+3}}$ denote the optimal rate of convergence for density estimation with $\tau+1$ continuous derivatives (Stone (1982)). Then it is well-known that

$$\sup_{p\in[\underline{P}+\delta,\overline{P}-\delta]}\left|\widehat{f}\left(p\right)-f\left(p\right)\right|=O\left(\eta_{M}^{*}\right)\ \text{ a.s.,} \tag{14}$$

for any positive $\delta$.

We summarize the convergence rates of $\widehat{\mathbf{q}}$, $\widehat{F}\left(\cdot\right)$, and $\widehat{f}\left(\cdot\right)$ in a proposition.

Proposition 3. *Suppose Assumptions D and R hold. Then for the estimators* defined *in (11) to (13):*

*(a)* $\sup_{p\in\left[\underline{P},\overline{P}\right]}\left|\widehat{F}\left(p\right)-F\left(p\right)\right|=O_{p}\left(1/\sqrt{M}\right)$;

*(b)* $\left\|\widehat{\mathbf{q}}-\mathbf{q}\right\|=O_{p}\left(1/\sqrt{M}\right)$;

*(c) For any positive* $\delta$, $\sup_{p\in[\underline{P}+\delta,\overline{P}-\delta]}\left|\widehat{f}\left(p\right)-f\left(p\right)\right|=O\left(\eta_{M}^{*}\right)$ *a.s.*

We next proceed to estimate $h\left(\cdot\right)$ using the estimators for $\mathbf{q}$, $f\left(\cdot\right)$ and $F\left(\cdot\right)$ described above.

**An estimator for** $h\left(\cdot\right)$

We start by obtaining an estimator for $R_{im}$, using

$$\widehat{R}_{im}=\begin{cases} P_{im}-\dfrac{\sum_{k=1}^{I}\widehat{q}_{k}k\left(1-\widehat{F}(P_{im})\right)^{k-1}}{\widehat{f}(P_{im})\sum_{k=1}^{I}\widehat{q}_{k}k(k-1)\left(1-\widehat{F}(P_{im})\right)^{k-2}} & \text{for } P_{im}\in[\underline{P}+\delta,\overline{P}-\delta] \\ +\infty & \text{otherwise}\end{cases}. \tag{15}$$

When $\widehat{R}_{im}<\infty$, $\widehat{R}_{im}$ is the estimator of $R_{im}$ based on on the feasible version of (10). In this case, $\widehat{R}_{im}$ is a smooth function of $\widehat{\mathbf{q}}$, $\widehat{F}\left(P_{im}\right)$ and $\widehat{f}\left(P_{im}\right)$. Lemma 6 shows the convergence rate of $\widehat{R}_{im}$ is determined by $\sup_{p\in[\underline{P}+\delta,\overline{P}-\delta]}\left|\widehat{f}\left(p\right)-f\left(p\right)\right|$. We effectively omit $\widehat{R}_{im}$ when $P_{im}\notin[\underline{P}+\delta,\overline{P}-\delta]$ for the purpose of estimating $h\left(\cdot\right)$. The omission does not prevent us from attaining the desired convergence rates because the probability that $\widehat{R}_{im}=+\infty$ is zero asymptotically.

Lemma 6. *Suppose Assumptions D and R hold. Then,*

$$\sup_{i,m\text{ s.t. }\widehat{R}_{im}<\infty}\left|\widehat{R}_{im}-R_{im}\right|=O\left(\eta_{M}^{*}\right)\ \ a.s.$$

We define our estimator for $h\left(\cdot\right)$ as follows:

$$\widehat{h}\left(r\right)=\frac{1}{MIb_{h,M}}\sum_{m=1}^{M}\sum_{i=1}^{I}K\left(\frac{\widehat{R}_{im}-r}{b_{h,M}}\right)\quad\text{for any } r, \tag{16}$$

where $K(\cdot)$ is a kernel function $b_{h,M}$ is the bandwidth. Under the conditions of Theorem 2, the uniform convergence rate of $\widehat{h}(\cdot)$ is determined by the convergence rate of $\widetilde{R}_{im}$.

THEOREM 2. *Suppose Assumptions D and R hold. Assume the following properties for components in (16):*
*(i) $K(\cdot)$ be a symmetric $\tau-$th order kernel with support $[-1,1]$;*
*(ii) $K(\cdot)$ is twice continuously differentiable on $[-1,1]$;*
*(iii) $b_{h,M}$ is proportional to $\left(\frac{\log M}{M}\right)^{\frac{1}{2\tau+3}}$.*
*Then for any $\varsigma > 0$, there exists $\delta > 0$ so that part (c) of Proposition 3 holds such that*

$$\sup_{r \in \left[\underline{R}+\varsigma, \overline{R}-\varsigma\right]} \left|\widehat{h}(r) - h(r)\right| = O\left(\left(\frac{\log M}{M}\right)^{\frac{\tau}{2\tau+3}}\right) \quad a.s.$$

The rate $\left(\frac{\log M}{M}\right)^{\frac{\tau}{2\tau+3}}$ is equal to $\frac{\eta_M^*}{b_{h,M}}$, which is the optimal convergence rate GPV derived in their paper. This rate is achieved by choosing $b_{h,M}$ that oversmooths relative to the optimal bandwidth for a $\tau-$times continuously differentiable density function.

Next, we extend the study of uniform convergence rate for an estimator of $h(\cdot)$ over $\left[\underline{R} + \varsigma_M, \overline{R} - \varsigma_M\right]$ for some $\varsigma_M = o(1)$. This requires us to provide a rate for an estimator of $f(\cdot)$ over intervals that expand towards $\left[\underline{P}, \overline{P}\right]$, which turns out to be an unusual problem[6] because $f(\cdot)$ has a pole at $\overline{P}$ (Lemma 4(b)). A slower convergence rate is expected in the presence of the pole, e.g., as the asymptotic variance and bias of kernel density estimator are respectively pointwise proportional to the underlying density and its derivatives respectively.

We begin by constructing an estimator for $f(\cdot)$ that can attain any convergence rate slower than $\eta_M^*$ over a suitably expanding support (cf. (14)), which allows us to estimate $h(\cdot)$ at any rate slower than $\left(\frac{\log M}{M}\right)^{\frac{\tau}{2\tau+3}}$ over an expanding support (cf. Theorem 2). We use the log-transformation approach suggested in Srisuma (2023) to estimate $f(\cdot)$. The novelty of his approach is, without having to specify the divergence rate of $f(\cdot)$, the transformed variable has a bounded density as long as $f(\cdot)$ satisfies a mild regularity condition. The idea is to then use the back-transformed estimator to quantify the convergence rate.[7]

Let $P_{im}^{\dagger} \equiv -\ln\left(\overline{P} - P_{im}\right)$. Denote the pdf of $P_{im}^{\dagger}$ by $f^{\dagger}(\cdot)$ that is positive on $[-\ln\left(\overline{P} - \underline{P}\right), \infty)$. By a change of variable, we have $f(p) = \frac{f^{\dagger}\left(-\ln\left(\overline{P}-p\right)\right)}{\overline{P}-p}$ for $p \in [\underline{P}, \overline{P}]$. Lemma 7 describes the notion

[6] One of the standard assumptions used in deriving uniform convergence rate of a kernel density estimator is boundedness of the underlying density (e.g., see Andrews (1995), Masry (1996), Fan and Yao (2003), Hansen (2008)).
[7] Srisuma (2023) also discusses other estimators that can obtain similar uniform convergence rates as the one based on a log-transformation.

of regularity for $f(\cdot)$ in terms of the existence of a limit at $\overline{P}$[8], which implies $f^{\dagger}(\cdot)$ and its derivatives are bounded (Lemma 8). We can then estimate $f^{\dagger}(\cdot)$ using a standard kernel density estimator at a $\eta_M^*-$convergence rate. We multiply this estimator by $\left(\overline{P}-p\right)^{-1}$ to get the back-transformed estimator. The back-transformation slows down the convergence rate, which we can control through the expansion rate of the support.

LEMMA 7. *Suppose Assumption D(i) holds. Then* $\lim_{p\to\overline{P}}\left(\overline{P}-p\right)f(p)$ *exists.*

LEMMA 8. *Suppose Assumptions D(i) and R hold. Then*

*(a)* $f^{\dagger}(\cdot)$ *is bounded;*

*(b)* $f^{\dagger}(\cdot)$ *admits upto* $\tau+1$ *continuous and bounded derivatives on* $[-\ln\left(\overline{P}-\underline{P}\right),\infty)$ *for the same* $\tau$ *as in Assumption R.*

Formally, the back-transformed estimator is defined as follows:

$$\begin{aligned} \widetilde{f}(p) &= \frac{\widehat{f}^{\dagger}\left(-\ln\left(\overline{P}-p\right)\right)}{\overline{P}-p}, \text{ where} \\ \widehat{f}^{\dagger}\left(p^{\dagger}\right) &= \frac{1}{MIb_{f^{\dagger},M}}\sum_{m=1}^{M}\sum_{i=1}^{I}K\left(\frac{P_{im}^{\dagger}-p^{\dagger}}{b_{f^{\dagger},M}}\right) \quad \text{for all } p^{\dagger}. \end{aligned} \tag{17}$$

By using a $(\tau+1)-$th higher order kernel and set $b_{f^{\dagger},M}$ to be proportional to $\left(\frac{\log M}{M}\right)^{\frac{1}{2\tau+3}}$, we have $\left|\widehat{f}^{\dagger}\left(p^{\dagger}\right)-f^{\dagger}\left(p^{\dagger}\right)\right|=O\left(\eta_M^*\right)$ a.s. uniformly over any fixed inner proper subset of $[-\ln\left(\overline{P}-\underline{P}\right),\infty)$. Since $\widetilde{f}(p)-f(p)=\frac{\widehat{f}^{\dagger}\left(-\ln\left(\overline{P}-p\right)\right)-f^{\dagger}\left(-\ln\left(\overline{P}-p\right)\right)}{\overline{P}-p}$, we have $\left|\widetilde{f}(p)-f(p)\right|=O\left(\eta_M^*\right)$ a.s. uniformly over any fixed inner subset of $\left[\underline{P},\overline{P}\right]$.

There are two factors that affect the uniform convergence rate of $\widetilde{f}(\cdot)$ over $[\underline{P}+\delta_M',\overline{P}-\delta_M'']$ when both $\delta_M'$ and $\delta_M''$ are $o(1)$. One is the boundary bias from estimating $f^{\dagger}(p)$ when $p^{\dagger}$ lies within a $b_{f^{\dagger},M}$-neighborhood from $-\ln\left(\overline{P}-\underline{P}\right)$. The other is the divergence of $\left(\overline{P}-p\right)^{-1}$ diverges as $p\to\overline{P}$. We can avoid the boundary effect at the lower boundary by limiting the rate $\delta_M'$ goes to zero according to $\overline{P}-\exp\left(\ln\left(\overline{P}-\underline{P}\right)-b_{f^{\dagger},M}\right)=o(\delta_M')$. We can choose the divergence rate from the back-transformation. Suppose $p\to\overline{P}$ at the rate $\delta_M''$, then

$$\sup_{p\in[\underline{P}+\delta_M',\overline{P}-\delta_M'']}\left|\widetilde{f}(p)-f(p)\right|=O\left(\frac{\eta_M^*}{\delta_M''}\right) \text{ a.s.}$$

[8]The proof of Lemma 7 makes clear that $f(\cdot)$ will be regular at $\overline{P}$ if $h(\cdot)$ is regular at $\overline{R}$ (i.e., $\lim_{r\to\overline{R}}\left(\overline{R}-r\right)h(p)$ exists). Assumption D(i) assumes $h(\cdot)$ is bounded, therefore it is regular at $\overline{R}$. We can thus replace boundedness with existence of a limit instead without changing any results. We refer the reader to Srisuma (2023) for further discussions on this notion of regularity as well as primitive conditions that imply it.

Thus, we can always find an interval expanding to $[\underline{P}, \overline{P}]$ that $\widetilde{f}(\cdot) - f(\cdot)$ converges uniformly over at any rate that is slower than $\eta_M^*$. We state this as a proposition.

PROPOSITION 4. *Suppose Assumptions D(i) and R hold. Then for any sequence of positive reals* $\{\eta_m\}_{m=1}^M$ *that decreases to* 0 *such that* $\eta_M^* = o(\eta_M)$*, there exists some sequence* $\{\delta_m\}_{m=1}^M$ *that decreases to* 0 *such that* $\sup_{p \in [\underline{P}+\delta_M, \overline{P}-\delta_M]} \left| \widetilde{f}(p) - f(p) \right| = O(\eta_M)$ *a.s.*

We can then estimate $R_{im}$ on using $\widetilde{f}(\cdot)$, and use it to estimate $h(\cdot)$ as done previously in (15) and (16) respectively. Specifically, for any $\delta_M > 0$ let

$$\widetilde{R}_{im} = \begin{cases} P_{im} - \frac{\sum_{k=1}^{I} \widehat{q}_k k \left(1-\widehat{F}(P_{im})\right)^{k-1}}{\widetilde{f}(P_{im}) \sum_{k=1}^{I} \widehat{q}_k k(k-1)\left(1-\widehat{F}(P_{im})\right)^{k-2}} & \text{for } P_{im} \in [\underline{P}+\delta_M, \overline{P}-\delta_M] \\ +\infty & \text{otherwise} \end{cases}, \quad (18)$$

$$\widetilde{h}(r) = \frac{1}{MIb_{h,M}} \sum_{m=1}^{M} \sum_{i=1}^{I} K\left(\frac{\widetilde{R}_{im} - r}{b_{h,M}}\right) \quad \text{for any } r. \quad (19)$$

The following results are similar to Lemma 6 and Theorem 2. They differ in that the rates do not reach the optimal rate and they hold over a sequence of expanding intervals.

LEMMA 9. *Suppose Assumptions D and R hold. Then for any sequence of positive reals* $\{\eta_m\}_{m=1}^M$ *that decreases to* 0 *such that* $\eta_M^* = o(\eta_M)$*, there exists some sequence* $\{\delta_m\}_{m=1}^M$ *as described in Proposition 4 such that*

$$\sup_{i,m \text{ s.t. } \widetilde{R}_{im}<\infty} \left| \widetilde{R}_{im} - R_{im} \right| = O(\eta_M) \ \ a.s.$$

THEOREM 3. *Suppose Assumptions D and R hold. Assume the following properties for components in (19):*
*(i)* $K(\cdot)$ *be a symmetric* $\tau-$*th order kernel with support* $[-1,1]$*;*
*(ii)* $K(\cdot)$ *is twice continuously differentiable on* $[-1,1]$*;*
*(iii)* $b_{h,M}$ *is proportional to* $\left(\frac{\log M}{M}\right)^{\frac{1}{2\tau+3}}$.
*Then for any* $\eta_M$ *that satisfies* $\eta_M^* = o(\eta_M)$ *and* $\eta_M = O\left(b_{h,M}^2\right)$*, and for* $\varsigma_M$ *that decreases to zero such that* $b_{h,M} = o(\varsigma_M)$*,*

$$\sup_{r \in [\underline{R}+\varsigma_M, \overline{R}-\varsigma_M]} \left| \widetilde{h}(r) - h(r) \right| = O\left(\frac{\eta_M}{b_{h,M}}\right) \quad a.s.$$

The uniform convergence rate for $\widetilde{h}(\cdot)$ is derived over an expanding support that avoids the boundary effect as well as anticipating the pole effect. We highlight the condition Theorem 3 imposes, which is not required in Theorem 2, is $\eta_M = O\left(b_{h,M}^2\right)$. This is a mild condition to handle the pole. In particular, this condition is not restrictive when $\tau \geq 2$. To see this, suppose $\eta_M = \eta_M^* \phi_M$ for some $\phi_M$ with $\lim_{M\to\infty} \phi_M = \infty$, then $\eta_M = O\left(b_{h,M}^2\right)$ is equivalent to $\phi_M \left(\frac{\log M}{M}\right)^{\frac{\tau-1}{2\tau+3}} = O(1)$. Since we are only be interested in $\phi_M$ that diverges to infinity slowly, we can choose it to diverge at an arbitrarily slow rate.

# 5 Extensions

We consider two extensions of the nonparametric identification arguments presented in Section 3. The first allows products to be vertically differentiated that can be useful for modelling systematic price differences across firms. The second introduces an intermediary that can search on behalf of consumers at a fee. Our discussions in this section will focus on identification. We will show the estimation strategy and the results on convergence rates of developed in Section 4 are applicable to these settings.

## 5.1 Vertical Product Differentiation

Let firm $i$'s product is characterized by $\nu_i \in \mathbb{R}$, which is a measure of differentiated quality. The econometrician will observe market share and prices but not quality of the products. The main modelling assumption employed by Wildenbeest (2011), in a complete information model, is that the difference between quality and marginal cost is the same for all firms. A natural way to extend his idea to an incomplete information game is to put a *common distribution* around $\nu_i$ for all $i$. We will show a quasi-symmetric equilibrium, where optimal pricing strategies between firms differ only by the differences in their qualities, can then be characterized analogously to Theorem 1.

**Consumer's Best Responses**

Consumers now value products from different firms differently. The utility they derive from purchasing from seller $i$ is $U_i$. We assume,

$$U_i := \nu_0 + \nu_i - P_i, \tag{20}$$

where $\nu_0$ denotes the common value of the product, $\nu_i$ denotes the valuation of the differentiating component due to firm $i$, and $P_i$ denotes its corresponding price. One can, for example, attribute $\nu_i$ to physical quality or other experience associated with purchasing from firm $i$. A consumer with search cost $c$ faces the following decision problem:

$$\max_{1\leq k\leq I} \mathbb{E}_L\left[U_{(k:k)}\right] - \ ck.$$

A purchase is always made so that $\nu_0$ does not enter our analysis, just as it does not in the model with a homogeneous product. For the moment suppose firms set prices such that $\{U_i\}_{i=1}^{I}$ is a random sample. Then, for $k \geq 1$, let $U_{(k:k)}$ be the maximum of $k$ i.i.d. random variables of utilities and $\mathbb{E}_L[\cdot]$ denotes an expectation where the random utilities have distribution described by the cdf $L(\cdot)$.

We denote the expected marginal utility gain from a purchase when a consumer searches one more firm when she has already searched $k-1$ firms by,

$$\Upsilon_k(L) := \mathbb{E}_L\left[U_{(k:k)}\right] - \mathbb{E}_L\left[U_{(k-1:k-1)}\right]. \tag{21}$$

We set $U_{(0:0)}$ to be 0. The consumer's best response is to search once if $c > \Upsilon_1(L)$, and search $k > 1$ times if $\Upsilon_{k-1}(L) < c \leq \Upsilon_k(L)$. Analogous to the discussions in Section 2.1, $\Upsilon_k(L)$ is positive and strictly decreasing when the distribution of $U_i$ is non-degenerate.

**Firm's Best Responses**

We assume firm $i$'s production cost consists of a sum of deterministic (determined by quality) and random components:

$$R_i = \nu_i + R_{0i},$$

where $R_{0i}$ has cdf $H_0(\cdot)$ supported on $\mathcal{R}_0 \in \left[\underline{R}_0, \overline{R}_0\right]$ for some $\overline{R}_0 > \underline{R}_0 > 0$. We denote the support of $R_i$ by $\mathcal{R}_i := \left[\nu_i + \underline{R}_0, \nu_i + \overline{R}_0\right]$. We assume firm costs are independent draws to preserve the independent value environment. Subsequently $\{R_{0i}\}_{i=1}^{I}$ is an i.i.d. sequence of random variables.

We restrict our attention to quasi-symmetric pricing strategies where firms' strategies are affine translations from one another. Denote firm $i$'s pricing strategy by $\beta_i(\cdot;\mathbf{q}) : \mathcal{R}_i \to \mathcal{P}_i$, where $\mathcal{P}_i = \left[\nu_i + \underline{P}_0, \nu_i + \overline{P}_0\right]$ and $\beta_i(\cdot;\mathbf{q}) = \nu_i + \beta_0(\cdot;\mathbf{q})$ and $\beta_0(\cdot;\mathbf{q}) : \mathcal{R}_0 \to \mathcal{P}_0 = \left[\underline{P}_0, \overline{P}_0\right]$. We denote the valuation-cost markup by $X_i := \nu_i - R_i$. By construction $X_i = -R_{0i}$ and $\{X_i\}_{i=1}^{I}$ is i.i.d. across firms. Since $U_i = \nu_i - P_i$, we can equivalently study the firm $i$'s profit maximization problem where the firm sets the level of utility consumers would get from buying its product instead of setting prices. I.e., for any $x_i \in \left[-\overline{R}_0, -\underline{R}_0\right]$, consider

$$\max_u \Gamma(u, x_i;\mathbf{q}), \text{ where}$$

$$\Gamma(u, x_i;\mathbf{q}) = (x_i - u)\sum_{k=1}^{I} q_k \frac{k}{I}\mathbb{P}\left[U_{(k-1:k-1)} \leq u\right].$$

Suppose a solution to the maximization problem above exists and let $\mu(x_i;\mathbf{q}) := \arg\max_u \Gamma(u, x_i;\mathbf{q})$ for any $(x_i, \mathbf{q})$. We assume that $\mu(x_i;\mathbf{q})$ to be increasing in $x_i$ and satisfies the boundary condition that $\mu\left(-\overline{R}_0;\mathbf{q}\right) = \overline{R}_0$. Under this premise, we can apply the arguments used to obtain (4) to show that

$$\mu\left(x\left(r_{0i}\right);\mathbf{q}\right)=x\left(r_{0i}\right)-\frac{\sum_{k=1}^{I}q_{k}k\int_{s=r_{0i}}^{\overline{R}_{0}}\left(1-H_{0}\left(s\right)\right)^{k-1}ds}{\sum_{k=1}^{I}q_{k}k\left(1-H_{0}\left(r_{0i}\right)\right)^{k-1}}, \tag{22}$$

for any $r_{0i}\in\mathcal{R}_{0}$ and $x\left(r_{0i}\right):=-r_{0i}$. Therefore $\left\{\mu\left(x\left(R_{0i}\right);\mathbf{q}\right)\right\}_{i=1}^{I}$ is an i.i.d. sequence of random utilities that firms offer to the consumers upon drawing $\left\{R_{0i}\right\}_{i=1}^{I}$ as a best response given $\mathbf{q}$.

For any $r_{i}=\nu_{i}+r_{0i}$, since $\mu_{i}\left(x_{i}\left(r_{i}\right);\mathbf{q}\right)=\nu_{i}-\beta_{i}\left(r_{i};\mathbf{q}\right)$, it follows that $\beta_{i}\left(r_{i};\mathbf{q}\right)=\nu_{i}+\beta_{0}\left(r_{0i};\mathbf{q}\right)$ where,

$$\beta_{0}\left(r_{0i};\mathbf{q}\right)=r_{0i}+\frac{\sum_{k=1}^{I}q_{k}k\int_{s=r_{0i}}^{\overline{R}_{0}}\left(1-H_{0}\left(s\right)\right)^{k-1}ds}{\sum_{k=1}^{I}q_{k}k\left(1-H_{0}\left(r_{0i}\right)\right)^{k-1}}. \tag{23}$$

$\beta_{0}\left(\cdot;\mathbf{q}\right)$ has an identical structure to $\beta\left(\cdot;\mathbf{q}\right)$ as defined in (4). Therefore the properties of each firm's pricing strategy derived here are the same as that of the homogeneous product case other than being shifted by a constant $\nu_{i}$. In particular $\beta_{0}\left(\cdot;\mathbf{q}\right)$ is strictly increasing when $q_{1}<1$, and its inverse takes the same form as (10) in Lemma 5.

### Equilibrium

We define a quasi-symmetric equilibrium where players using pricing strategies that are affine translation from each other. Theorem 4 gives a characterization of the equilibrium (cf. Theorem 1). Its proof and the definition of a quasi-symmetric equilibrium are in the Appendix.

THEOREM 4. *In a quasi-symmetric equilibrium* $\left(\mathbf{q},F_{0}\right)\in\mathbb{S}^{I-1}\times\mathcal{F}_{0}$*, where the equilibrium pricing strategies are strictly increasing,* $\mathbf{q}$ *satisfies the following system of equations:*

$$q_{k}=\left\{\begin{array}{ll}1-G\left(\int F_{0}\left(p\right)\left(1-F_{0}\left(p\right)\right)dp\right) & \text{for } k=1\\ G\left(\int F_{0}\left(p\right)\left(1-F_{0}\left(p\right)\right)^{k-1}dp\right)-G\left(\int F_{0}\left(p\right)\left(1-F_{0}\left(p\right)\right)^{k}dp\right) & \text{for } 1<k<I\end{array}\right.,$$

*where* $F_{0}\left(p\right)=H_{0}\left(\xi_{0}\left(p;\mathbf{q}\right)\right)$ *for all* $p\in\left[\underline{P},\overline{P}\right]$ *and* $\xi_{0}\left(\cdot;\mathbf{q}\right)$ *is the inverse of* $\beta_{0}\left(\cdot;\mathbf{q}\right)$.

### 5.1.1 Identification

We assume our data satisfy the following conditions.

ASSUMPTION DE1. $\left\{\left(Y_{im},P_{im}\right)\right\}_{i=1,m=1}^{I,M}$ *is a sequence of random variables such that:*

*(i) there exists* $(\mathbf{q}, F_0) \in \mathbb{S}^{I-1} \times \mathcal{F}$ *with* $q_1 \in (0,1)$ *so that* $P_{im} = \nu_i + \beta_0(R_{0im}; \mathbf{q})$ *where* $\beta_0(\cdot; \mathbf{q})$ *has been defined in (23) and* $\{R_{0im}\}_{i=1,m=1}^{I,M}$ *is i.i.d. with positive and finite density almost everywhere on* $[\underline{R}_0, \overline{R}_0]$*;*

*(ii)* $\{(Y_{1m}, \ldots, Y_{Im})\}_{m=1}^{M}$ *is i.i.d. such that the joint distribution of* $(Y_{im}, P_{0im})$ *satisfies,*

$$\mathbb{E}[Y_{im}|P_{0im}] = \sum_{k=1}^{I} q_k \frac{k}{I} (1 - F_0(P_{0im}))^{k-1}. \tag{24}$$

Assumption DE1 has an analogous interpretations to Assumption D. If we observe $\{\nu_i\}_{i=1}^{I}$, we can construct $\{P_{0im}\}_{i=1,m=1}^{I,M}$, then identification immediately follows the same steps described in Section 3. In particular: (i) use $\{P_{0im}\}_{i=1,m=1}^{I,M}$ to identify $f_0(\cdot)$ and $F_0(\cdot)$; (ii) identify $\mathbf{q}$ from (24) (cf. Lemma 3), combine it with $\{\Upsilon_k\}_{k=1}^{I-1}$, we can identify $\{G(\Upsilon_k)\}_{k=1}^{I-1}$ (cf. Proposition 1); (iii) recover $\{R_{0im}\}_{i=1,m=1}^{I,M}$ from

$$R_{0im} = P_{0im} - \frac{\sum_{k=1}^{I} q_k k (1 - F_0(P_{0im}))^{k-1}}{f_0(P_{0im}) \sum_{k=2}^{I} q_k k (k-1) (1 - F_0(P_{0im}))^{k-2}}, \tag{25}$$

cf. (10), which in turn identifies $H_0(\cdot)$ (cf. Proposition 2).

We, however, do not observe $\{\nu_i\}_{i=1}^{I}$. The key insight to proceed is that optimal search behavior is determined by the shape of the equilibrium price distributions, which is the same for all firms, and not their locations that may differ. Subsequently, relative utilities are identified by relative demeaned prices. To see this recall $U_{im} = \nu_i - P_{im}$, so for all $i$ and $j$:

$$U_{im} - U_{jm} = P_{0jm} - P_{0im} = \omega_{jm} - \omega_{im}, \tag{26}$$

where $\omega_{im}$ denotes $P_{im} - \mathbb{E}[P_{im}]$, thus the second equality above follows from $\mathbb{E}[P_{0im}] = \mathbb{E}[P_{0jm}]$.

Our identification results rely on the distribution of $\omega_{im}$, which identified. We denote the pdf and cdf of $\omega_{im}$ by $w(\cdot)$ and $W(\cdot)$ respectively. Note that $F_0(\cdot)$ and $W(\cdot)$ are parallel to each other by construction. A useful relation that immediately follows from inspecting (26) is that the cdfs of $P_{0im}$ and $\omega_{im}$ coincide when evaluated at their respective points of realizations. We state this as a lemma. We use it to identify the consumer search distribution.

Lemma 10. *Suppose Assumption DE1 holds. Then* $F_0(P_{0im}) = W(\omega_{im})$ *for all* $i$ *and* $m$.

Proposition 5. *Suppose Assumption DE1 holds. Then* $G(\Upsilon_k)$ *is identified for* $k = 1, \ldots, I-1$.

Proof. By Lemma 10, any $\mathbf{q}$ that satisfies (24) also satisfies

$$\mathbb{E}[Y_{im}|\omega_{im}] = \sum_{k=1}^{I} q_k \frac{k}{I} (1 - W(\omega_{im}))^{k-1}.$$

We can then identify $\mathbf{q}$ in closed-form as done in Lemma 3. From (21), we can also identify $\Upsilon_k$ in the same way we identify $\Delta_k$ in Section 3.1 by replacing the raw prices with the demeaned prices. We can then apply the argument used to prove Proposition 1 to identify $\{G(\Upsilon_k)\}_{k=1}^{I-1}$ from $\mathbf{q}$ and $\{\Upsilon_k\}_{k=1}^{I-1}$. ■

On the supply side, we can identify the shape of the distribution of $R_{0im}$ but not its location. This is clear from (25) because we can only identify the shape of the distribution of $P_{0im}$. More precisely, what we can identify is the distribution of $\rho_{im} := R_{0im} - \mathbb{E}[P_{0im}]$.

PROPOSITION 6. *Suppose Assumption DE1 holds. Then the distribution of $\rho_{im}$ is identified.*
PROOF. Replace $(P_{0im}, f_0(P_{0im}), F_0(P_{0im}))$ in the RHS of (25) by $(\omega_{im}, w(\omega_{im}), W(\omega_{im}))$ to construct $\rho_{im}$. Then apply Lemma 10.■

Propositions 5 and 6 show we can use $\{\omega_{im}\}_{i=1,m=1}^{I,M}$ instead of price to identify the demand and supply side parameters in the same way as done in Sections 3.1 and 3.2 respectively. Analogous estimators and results on convergence rates discussed in Section 4 are therefore immediately applicable.

It is worth commenting that not knowing $\{\nu_i\}_{i=1}^{I}$ does not limit the scope of counterfactual studies relative to the model with homogenous goods. This is because consumers in our model bear the cost of quality differences and get compensated in equal amount in terms of utility. Thus we can study changes in search behavior and the price distribution associated with quality adjusted production costs. We can identify these effects by comparing the difference of price distributions generated from the old and new equilibria where firms are treated symmetrically such that every firm draws cost from the same distribution as $\rho_{im}$.

## 5.2 Intermediary

Next, we assume there is an intermediary, which we will also refer to as *broker*. Our model is closely related to Salz (2020). We adopt his key assumption that a consumer with very high cost consumers prefer to pay a broker to search. The broker then performs an exhaustive search and gives such consumer the lowest price.

### Consumer's Best Responses

Previously we saw that it is optimal for a consumer that draws $c > \Delta_1(F)$ to search once. If a broker charges $\phi$ to conduct an exhaustive search for such consumer, the cost that makes a consumer indifferent between searching once or delegating search solves: $\mathbb{E}_F[P] + c = \phi + \mathbb{E}_F\left[P_{(1:I)}\right]$. Let us denote such cost by $\Delta_0(F) := \phi + \mathbb{E}_F\left[P_{(1:I)}\right] - \mathbb{E}_F[P]$. We take $\phi$ to be exogenous. Following Salz

(2020), we focus on an equilibrium where $c > \Delta_0(F)$ has positive probability and $\Delta_0(F) > \Delta_1(F)$. This allows us to generalize Lemma 1 and define a consumer's best response to be a map $\widetilde{\sigma}_D : \mathcal{F} \to \mathbb{S}^I$ such that for any $F$ in $\mathcal{F}$,

$$\widetilde{\sigma}_D(F) = \begin{cases} 1 - G(\Delta_k(F)) & \text{for } k = 0 \\ G(\Delta_{k-1}(F)) - G(\Delta_k(F)) & \text{for } 0 < k < I \\ G(\Delta_k(F)) & \text{for } k = I \end{cases}.$$

**Firm's Best Response**

Let us denote the proportion of consumers that use a broker by $q_0 < 1$. Firm $i$ chooses price to maximize $\widetilde{\Lambda}(p, R_i; q_0, \mathbf{q})$ where:

$$\widetilde{\Lambda}(p, R_i; q_0, \mathbf{q}) := (1 - q_0)(p - R_i) \sum_{k=1}^{I} q_k \frac{k}{I} \mathbb{P}\left[P_{(1:k-1)} > p\right] + q_0 (p - R_i) \mathbb{P}\left[P_{(1:I-1)} > p\right],$$

where $\mathbf{q}$ now represents a vector of proportions of consumers that search a different number of firms, conditioning on them searching, so that $(q_0, \mathbf{q}) \in [0, 1) \times \mathbb{S}^{I-1}$. The two additive components on the RHS in the display above are firm $i$'s expected payoffs from the consumers that search and use an intermediary respectively. It will be useful to define the vector $\widetilde{\mathbf{q}}(q_0, \mathbf{q}) := (\widetilde{q}_k(q_0, \mathbf{q}))_{k=1}^{I} \in \mathbb{S}^{I-1}$ with the following components,

$$\widetilde{q}_k(q_0, \mathbf{q}) = (1 - q_0) q_k \quad \text{for} \quad 0 < k < I \quad \text{and} \quad \widetilde{q}_I(q_0, \mathbf{q}) = (1 - q_0) q_I + q_0. \tag{27}$$

Then, we have for all $(p, r)$ and $(q_0, \mathbf{q})$,

$$\widetilde{\Lambda}(p, r; q_0, \mathbf{q}) = (p - r) \sum_{k=1}^{I} \widetilde{q}_k(q_0, \mathbf{q}) \frac{k}{I} \mathbb{P}\left[P_{(1:k-1)} > p\right] = \Lambda(p, r; \widetilde{\mathbf{q}}(q_0, \mathbf{q})),$$

where $\Lambda(\cdot)$ is the same function used in Section 2.2. Subsequently, the solution to the maximization problem above is given by $\beta(\cdot; \widetilde{\mathbf{q}}(q_0, \mathbf{q}))$ (see (4)) where $\beta(\cdot; \widetilde{\mathbf{q}}(q_0, \mathbf{q}))$ is strictly increasing when $\widetilde{q}_1(q_0, \mathbf{q}) < 1$. Notably, its inverse takes the same form as (10) given in Lemma 5 and the characteristics of the equilibrium price distribution of Lemma 7 applies when $\widetilde{q}_1(q_0, \mathbf{q}) > 0$.

**Equilibrium**

We characterize an equilibrium for a search model with an intermediary as follows.

THEOREM 5. *In a symmetric equilibrium* $(q_0, \mathbf{q}, F) \in [0, 1) \times \mathbb{S}^{I-1} \times \mathcal{F}$, *where* $\Delta_0(F) > \Delta_1(F)$ *and the equilibrium pricing strategies are strictly increasing,* $(q_0, \mathbf{q})$ *satisfies the following system of* *equations:*

$$\begin{array}{rcl} q_0 & = & 1 - G(\Delta_k(F)) \quad \text{for } k = 0 \\ (1-q_0)\, q_k & = & G(\Delta_{k-1}(F)) - G(\Delta_k(F)) \quad \text{for } 0 < k < I \\ (1-q_0)\, q_I & = & G(\Delta_k(F)) \quad \text{for } k = I \end{array},$$

*where* $F(p) = H(\xi(p; \widetilde{\mathbf{q}}(q_0, \mathbf{q})))$ *for all* $p \in [\underline{P}, \overline{P}]$ *and* $\xi(\cdot; \widetilde{\mathbf{q}}(q_0, \mathbf{q}))$ *is the inverse of* $\beta(\cdot; \widetilde{\mathbf{q}}(q_0, \mathbf{q}))$.

**Identification**

We can apply the same identification strategy as in Section 3 when $q_0$ is known. We make the following assumptions.

Assumption DE2. $\{(Y_{im}, P_{im})\}_{i=1,m=1}^{I,M}$ *is a sequence of random variables such that for a known* $q_0 \in (0,1)$*:*

*(i) there exists* $(\mathbf{q}, F) \in \mathbb{S}^{I-1} \times \mathcal{F}$ *with* $q_1 \in (0,1)$ *so that* $P_{im} = \beta(R_{im}; \widetilde{\mathbf{q}}(q_0, \mathbf{q}))$ *where* $\beta(\cdot; \mathbf{q})$ *and* $\widetilde{\mathbf{q}}(q_0, \mathbf{q})$ *are defined in (4) and (27) respectively, and* $\{R_{im}\}_{i=1,m=1}^{I,M}$ *is i.i.d. with positive and finite density almost everywhere on* $[\underline{R}, \overline{R}]$*;*

*(ii)* $\{(Y_{1m}, \ldots, Y_{Im})\}_{m=1}^{M}$ *is i.i.d. such that the joint distribution of* $(Y_{im}, P_{im})$ *satisfies,*

$$\mathbb{E}[Y_{im}|P_{im}] = \sum_{k=1}^{I} \widetilde{q}_k(q_0, \mathbf{q}) \frac{k}{I} (1 - F(P_{im}))^{k-1}. \tag{28}$$

The known $q_0$ assumption is suitable when the proportion of consumers who used a broker can be identified directly from the data. Examples of this include Salz (2020) and our application in Section 7. Other than assuming $q_0 \in (0,1)$, the discussions on Assumption D are applicable to the remainder of Assumption DE2.

Under Assumption DE2, $\widetilde{\mathbf{q}}(q_0, \mathbf{q})$ is identified as long as $\mathbb{E}[X_{im} X_{im}^\top]$ has full rank ($X_{im}$ is defined as in Section 3.1). We can then identify $\{G(\Delta_k)\}_{k=0}^{I-1}$ as in Proposition 1.

Proposition 7. *Suppose Assumption DE2 holds and* $\mathbb{E}[X_{im} X_{im}^\top]$ *has full rank. Then* $G(\Delta_k)$ *is identified for* $k = 0, 1, \ldots, I-1$.

Proof. First apply Lemma 3 to identify $\widetilde{\mathbf{q}}(q_0, \mathbf{q})$. Since $q_0$ is known, $\mathbf{q}$ is identified from $\widetilde{\mathbf{q}}(q_0, \mathbf{q})$. Subsequently, $G(\Delta_0) = 1 - q_0$ and $G(\Delta_k) = 1 - q_0 - (1 - q_0) \sum_{k'=1}^{k} q_{k'}$ for $k > 0$.$\blacksquare$

Once $\widetilde{\mathbf{q}}(q_0, \mathbf{q})$ is known, we can identify $H(\cdot)$. The argument in Section 3.2 applies directly, because the inverse of $\beta(\cdot; \widetilde{\mathbf{q}}(q_0, \mathbf{q}))$ takes the same form as (10) and the price distribution is identified.

# 6 Monte Carlo Study

The purpose of this section is to numerically investigate theoretical features of our model. We consider a simple design with three firms. Consumers draw costs from a distribution with cdf $G(c) = \sqrt{c}$ for $c \in [0,1]$. Firms draw costs from a uniform distribution on $[0,1]$. We solved for the equilibrium of the game by iterating the system of equations in (6). We tried different initial values and found only one equilibrium that generates price dispersion with $\mathbf{q} = (0.7852, 0.0455, 0.1693)$. We generate data from this equilibrium for 333 markets, so $IM = 999$, by drawing prices prices from (4) and market shares from (7).

We focus on the nonparametric estimators of $f(\cdot)$ and $h(\cdot)$. We estimate $F(\cdot)$ and $\mathbf{q}$ using the estimators described in Section 4. For $f(\cdot)$ and $h(\cdot)$, while the estimators mentioned in Section 4 are sufficient in delivering the desired convergence rate uniformly over an expanding interval, in practice we can use data outside of the interval that are closer to the boundaries if they can be well estimated. Following the suggestion of Hickman and Hubbard (2015), who estimated a first-price auction model, we employ a boundary corrected kernel and use all observations. Their choice for the boundary correction is based on the estimator of Karunamuni and Zhang (2008, henceforth KZ), and they show it works well in small samples (also see Li and Liu (2015) in another auction application). We note that our estimation problem is more challenging than a pure auction setup because we have to estimate $f(\cdot)$, which has a pole, and the estimation of $h(\cdot)$ has additional sampling errors from estimating $\mathbf{q}$ and $F(\cdot)$.

We consider two estimators for $f(\cdot)$. $\widehat{f}_1(\cdot)$ is an estimator based on KZ that accounts for the boundary effects but ignores the presence of the pole. $\widehat{f}_2(\cdot)$ uses the transformation described in equation (17) to accommodate the pole and applies boundary correction at the lower boundary. We use the Epanechnikov kernel for all of our estimators. Boundary correction uses the optimal endpoint kernel and associated plug-in constants and bandwidths suggested in KZ. Figures 1 and 2 plot the mean and the 5th and 95th percentiles for each $\left(\widehat{f}_1(\cdot), \widehat{f}_2(\cdot)\right)$ against the true price pdf. We see that $\widehat{f}_1(\cdot)$ performs quite well near the lower support point but not near the pole. $\widehat{f}_2(\cdot)$ performs much better near the pole. A careful inspection, however, shows the bias of $\widehat{f}_2(\cdot)$ is generally larger than that of $\widehat{f}_1(\cdot)$ away from the pole.We next estimate $h(\cdot)$. The plots in Figures 3 and 4 contain the mean and the 5th and 95th percentiles of KZ boundary corrected estimators, $\left(\widehat{h}_1(\cdot), \widehat{h}_2(\cdot)\right)$, that correspond respectively to $\left(\widehat{f}_1(\cdot), \widehat{f}_2(\cdot)\right)$. These figures also include analogous plots from an infeasible KZ boundary corrected estimator constructed from the estimated costs when the true $f(\cdot)$ is used, while $\mathbf{q}$ and $F(\cdot)$ are still estimated, to highlight the effect of density estimation. The infeasible estimator is generally the superior estimator as expected, although it is worth noting that even the infeasible estimator still suffer from the boundary effect. For the feasible estimators, $\widehat{h}_2(\cdot)$

Figure 1: $\widehat{f}_1(\cdot)$ - KZ boundary correction

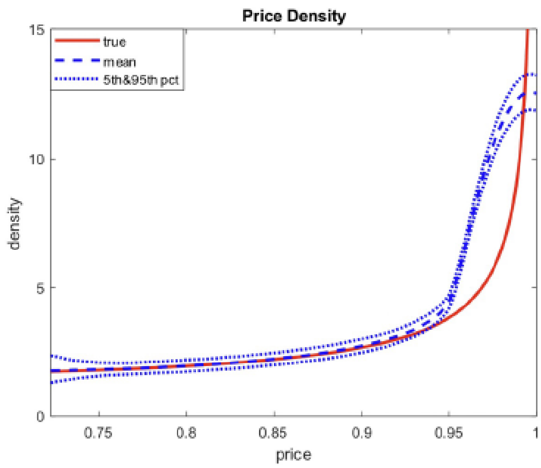
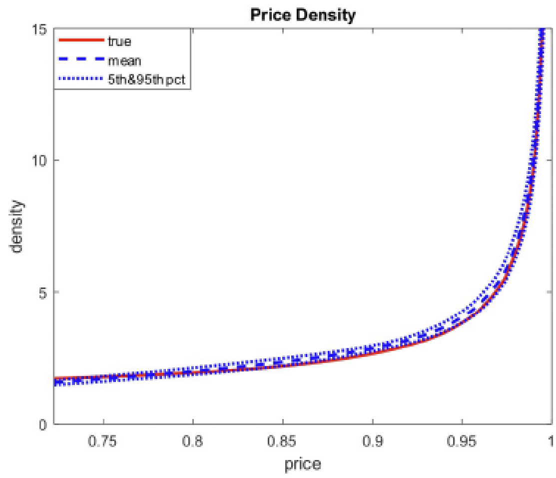



Figure 2: $\widehat{f}_2(\cdot)$ - Transformation and KZ boundary correction




has lower bias over its lower half of the support compared to the upper half due to substantial bias from estimating $f(\cdot)$ near the pole. In contrast, $\widehat{h}_2(\cdot)$ performs extremely well closer to the upper support and its distribution is concentrated around the mean over the whole support, however its bias increases as it approaches the lower boundary.

The simulation study illustrates the performance of $\widehat{h}_j(\cdot)$ inherits characteristics of $\widehat{f}_j(\cdot)$. Therefore it is clear one should account for the boundary effect at the lower support as well as the pole at the upper support. One way to proceed in practice is to perform some kind of model averaging. Asymptotically, such estimator will be consistent since $\left(\widehat{f}_1(\cdot), \widehat{f}_2(\cdot)\right)$, and subsequently $\left(\widehat{h}_1(\cdot), \widehat{h}_2(\cdot)\right)$, are consistent estimators on the interior of the support.

# 7 Empirical Application

## 7.1 Data and setup

Our application is related to our earlier work estimating the value of information provided by intermediaries in the UK mortgage market[9]. Our aim here is to illustrate how adaptations of our baseline model can be applied to a rich dataset through the lens of a search model. The particular adaptations are: (i) incorporate observable product and consumer heterogeneity; (ii) introduce an intermediary who provides information about prices in exchange for a fee (see section 5.2); (iii) show how some parts of the estimation procedure can be conducted parametrically.

[9]See the technical report in Myśliwski and Rostom (2022). To keep this section succinct and self-contained, we further refer the reader to that paper for a detailed description of the market, dataset, descriptive evidence of price dispersion, importance of intermediaries and counterfactuals.

Figure 3: $\widehat{h}_1(\cdot)$ - Based on $\widehat{f}_1(\cdot)$

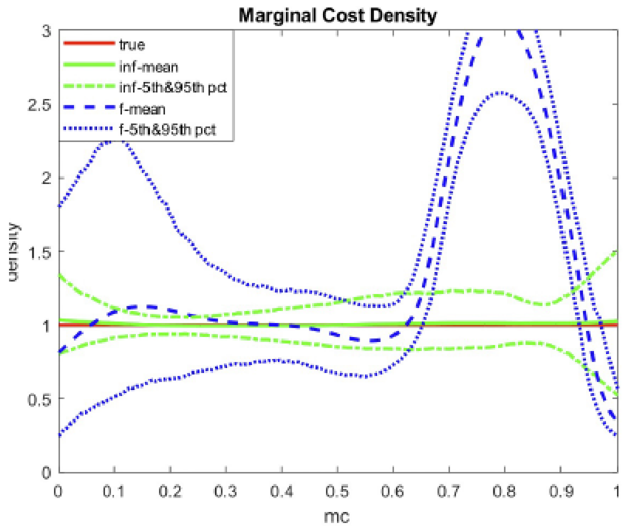
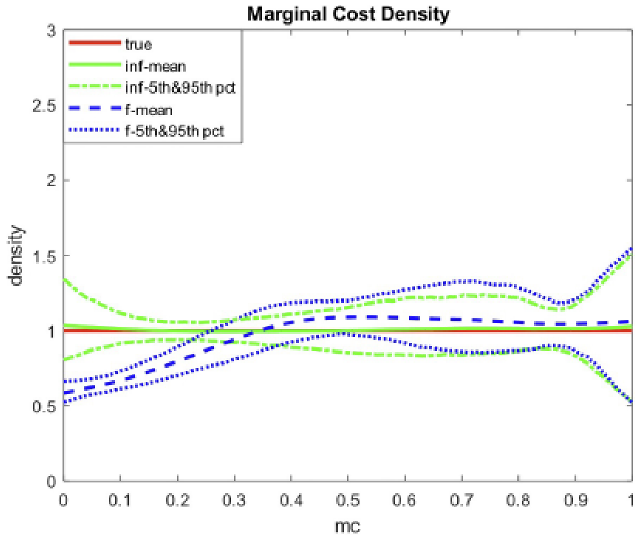



Figure 4: $\widehat{h}_2(\cdot)$ - Based on $\widehat{f}_2(\cdot)$




The data come from Product Sales Database containing loan-level administrative data for all new mortgages in the UK. We focus on fixed rate mortgage products with two-, three-, and five-year durations; and to loan sizes less than £1M from 2016 and 2017. The empirical exercise presented here differs from Myśliwski and Rostom (2022, MR hereafter) in several respects. To reduce dimensionality of the observed characteristics, we use a subset of the data covering only regular loan types (i.e. those without flexible repayment schemes and cashback options). Secondly, we focus on competition between four, instead of six biggest lenders. Our final dataset contains information on over 700,000 individual transactions, which are then aggregated up to monthly prices and market shares of each of the four banks conditional on loan and borrower characteristics. Finally, we use parametric estimators for price and marginal cost distributions.

We follow Allen, Clark, and Houde (2019)) and construct price based on monthly mortgage cost defined as $P = iL + \frac{Fee}{N}$, where $i$ is the interest rate in the initial, fixed-rate period, $L$ is the size of the loan, $Fee$ is the up-front fee, and $N$ is the initial period of the mortgage contract (24, 36, or 60 months). We detrend prices to remove dispersion from macroeconomic shocks (e.g. changes to the Bank of England's interest rate) and deflate them to January 2016 levels. We further normalize this measure to correspond to the median-sized loan in the data (around £150k). Even with the detrending and normalization, loans can still differ of observable characteristics which we label as $z_l$. Moreover, we allow for different borrower types (e.g. younger/older, more/less wealthy households) to draw their search costs from different distributions. Those characteristics are denoted by $z_b$. Table

1 lists out $z_b$ and $z_l$.

| Variable | Values |
|---|---|
| | Borrower characteristics: $z_b$ |
| Age | $< 30, 30+$ |
| Income | Below/above median |
| FTB status | First-time-buyer (FTB), Non-FTB |
| Location | Urban, Rural |
| | Loan characteristics: $z_l$ |
| LTV | $\leq 70$, 71-75, 76-80, 81-85, 86-90, 91-95 |
| Deal length | 2-, 3-, 5-year |
| Term | $< 10, (10, 15], (15, 20], (20, 25], (25, 30], (30, 35]$ |
| Loan value | 4 quartiles |

Table 1. List of observable heterogeneities in the model.

## 7.2 Estimation

We see borrowers with different prices for the same borrower-loan type $(z_b, z_l)$. To construct an estimator based on our identification strategy in the previous sections, we aggregate consumers' prices in each market. In what follows, let $Y_{im}^{(z_b,z_l)}$ and $P_{im}^{(z_b,z_l)}$ denote market share and price of lender $i$ in month $m$ for any $(z_b, z_l)$. We use the individual median price to represent $P_{im}^{(z_b,z_l)}$. We believe such aggregation rule is reasonable, following Benetton (2021), who argued the UK mortgage market is characterized by posted rather than individualized prices since mortgage offers are based on observables with little room for negotiation. Variation in prices for each lender-market-product combination is therefore due to discretization of loan characteristics. We have explored other representative market prices as a robustness check. Our results are largely robust to different representations for $P_{im}^{(z_b,z_l)}$ (we used mean or several other quantiles).

We estimate our model parametrically, following these steps:

1. *Estimate* $F(\cdot|z_b, z_l)$. Take $\left\{P_{im}^{(z_b,z_l)}\right\}_{i=1,m=1}^{I,M}$ to be a random sample from a Beta distribution conditional on $(z_b, z_l)$. We parameterized the distribution so the two shape parameters are governed by $\theta(z_b, z_l) = \left(\theta_{11}^\top z_b + \theta_{21}^\top z_l, \theta_{12}^\top z_b + \theta_{22}^\top z_l\right)$. We pooled prices across $(z_b, z_l)$ and estimated $\left(\theta_{11}^\top, \theta_{12}^\top, \theta_{21}^\top, \theta_{22}^\top\right)^\top$ by maximum likelihood[10].

[10] We used the `betareg` command on R. We first rescaled price onto $[0, 1]$. We recovered the original pdf and cdf

2. *Estimate* $\widetilde{\mathbf{q}}(z_b, z_l)$*:* Use $\left\{\left(Y_{im}^{(z_b,z_l)}, P_{im}^{(z_b,z_l)}\right)\right\}_{i=1,m=1}^{I,M}$ to perform constrained least squares based on minimizing $\left\|\mathbf{Y}^{(z_b,z_l)} - \widehat{\mathbf{X}}^{(z_b,z_l)}q\right\|^2$ subject to $q$ summing to 1 and $q \geq 0$ to obtain $\widehat{\widetilde{\mathbf{q}}}(z_b, z_l)$. $\mathbf{Y}^{(z_b,z_l)}$ denotes the vector of market shares and $\left(\widehat{\mathbf{X}}_m^{(z_b,z_l)}\right)_{ik} = \frac{k}{I}\left(1 - \widehat{F}\left(P_{im}|z_b, z_l\right)\right)^{k-1}$ with $\widehat{F}(\cdot|z_b, z_l)$ taken from Step 1.

3. *Estimate* $G(\Delta_k(z_b, z_l)|z_b)$*.* Estimate $G(\Delta_0(z_b, z_l)|z_b)$ by $1-\widehat{q}_0(z_b, z_l)$ and $G(\Delta_k(z_b, z_l)|z_b)$ by $1-\widehat{q}_0(z_b, z_l)-(1-\widehat{q}_0(z_b, z_l))\sum_{k'=1}^{k}\widehat{q}_{k'}(z_b, z_l)$ for $k > 0$. $\widehat{q}_0(z_b, z_l)$ is the proportion of borrowers using a broker and $\widehat{q}_k(z_b, z_l) = \frac{\left(\widehat{\widetilde{\mathbf{q}}}(z_b,z_l)\right)_k}{1-\widehat{q}_0(z_b,z_l)}$ for $1 \leq k < I$ and $q_I(z_b, z_l) = \frac{\left(\widehat{\widetilde{\mathbf{q}}}(z_b,z_l)\right)_I - \widehat{q}_0(z_b,z_l)}{1-\widehat{q}_0(z_b,z_l)}$ where $\widehat{\widetilde{\mathbf{q}}}(z_b, z_l)$ is from Step 2.

4. *Estimate* $\Delta_k(z_b, z_l)$*.* Use $\widehat{F}(\cdot|z_b, z_l)$ from Step 1 to simulate prices and estimate $\Delta_k(z_b, z_l)$ by $\mathbb{E}_{\widehat{F}(\cdot|z_b,z_l)}\left[P_{(1:k)}\right] - \mathbb{E}_{\widehat{F}(\cdot|z_b,z_l)}\left[P_{(1:k+1)}\right]$ for $k > 0$ and $\Delta_0(z_b, z_l)$ by $\widehat{\phi}(z_b, z_l) + \mathbb{E}_{\widehat{F}(\cdot|z_b,z_l)}\left[P_{(1:I)}\right] - \mathbb{E}_{\widehat{F}(\cdot|z_b,z_l)}[P]$ where $\widehat{\phi}(z_b, z_l)$ is the median broker fee.

5. *Estimate* $h(\cdot|z_l)$*.* Construct $\widehat{R}_{im}^{(z_b,z_l)}$ by $P_{im}^{(z_b,z_l)} - \frac{\sum_{k=1}^{I}\left(\widehat{\widetilde{\mathbf{q}}}(z_b,z_l)\right)_k k\left(1-\widehat{F}\left(P_{im}^{(z_b,z_l)}|z_b,z_l\right)\right)^{k-1}}{\widehat{f}\left(P_{im}^{(z_b,z_l)}|z_b,z_l\right)\sum_{k=1}^{I}\left(\widehat{\widetilde{\mathbf{q}}}(z_b,z_l)\right)_k k(k-1)\left(1-\widehat{F}\left(P_{im}^{(z_b,z_l)}|z_b,z_l\right)\right)^{k-2}}$ where $\left(\widehat{f}(\cdot|z_b, z_l), \widehat{F}(\cdot|z_b, z_l)\right)$ are from Step 1 and $\widehat{\widetilde{\mathbf{q}}}(z_b, z_l)$ is from Step 2. Take $\left\{\widehat{R}_{im}^{(z_b,z_l)}\right\}_{i=1,m=1}^{I,M}$ to be a random sample from Beta distribution conditional with parameters are governed by $\pi(z_l) = \left(\pi_1^\top z_l, \pi_2^\top z_l\right)$.

Other than the parametric estimation of price and lender's cost distributions, we follow closely the nonparametric identification strategy from Step 2 to the construction of $\widehat{R}_{im}^{(z_b,z_l)}$ in Step 5. In particular, we do not specify $G(\cdot|z_b)$ to come from a particular parametric distribution. We choose the Beta distribution because it allows for poles at the boundaries. Lastly, relating to our discussion in Section 3.2, variation in $z_l$ allows us to identify $G(\cdot|z_b)$ at more points than the case without lender's observed heterogeneity. Specifically, for a given $z_b$, we have estimates of $G(\Delta_k(z_b, z_l)|z_b)$ for $k = 1, 2, 3$ for each loan characteristics combination $z_l$. Following Sanches et al. (2018), for example, we can pool estimates of $\{G(\Delta_k(z_b, z_l)|z_b)\}_{k=1}^{3}$ across all loan characteristics combinations by series estimation.

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by: (i) let $\widetilde{P}_{im}^{(z_b,z_l)} = \frac{P_{im}^{(z_b,z_l)} - \underline{P}^{(z_b,z_l)}}{\overline{P}^{(z_b,z_l)} - \underline{P}^{(z_b,z_l)}}$ where $\left(\underline{P}^{(z_b,z_l)}, \overline{P}^{(z_b,z_l)}\right)$ are the min and max prices; (ii) perform betareg with $\left\{\widetilde{P}_{im}^{(z_b,z_l)}\right\}_{i=1,m=1}^{I,M}$ to estimate $\widetilde{f}(\cdot|z_b, z_l)$ and $\widetilde{F}(\cdot|z_b, z_l)$; (iii) estimate $f(\cdot|z_b, z_l)$ and $F(\cdot|z_b, z_l)$ using $f(p|z_b, z_l) = \frac{1}{\overline{P}^{(z_b,z_l)} - \underline{P}^{(z_b,z_l)}}\widetilde{f}\left(\frac{p-\underline{P}^{(z_b,z_l)}}{\overline{P}^{(z_b,z_l)} - \underline{P}^{(z_b,z_l)}}|z_b, z_l\right)$ and $F(p|z_b, z_l) = \widetilde{F}\left(\frac{p-\underline{P}^{(z_b,z_l)}}{\overline{P}^{(z_b,z_l)} - \underline{P}^{(z_b,z_l)}}|z_b, z_l\right)$ respectively.

### 7.3 Results

For brevity's sake, we present a selection of estimation results illustrating the main advantages of our approach. First, we summarize quartiles of search cost distributions.

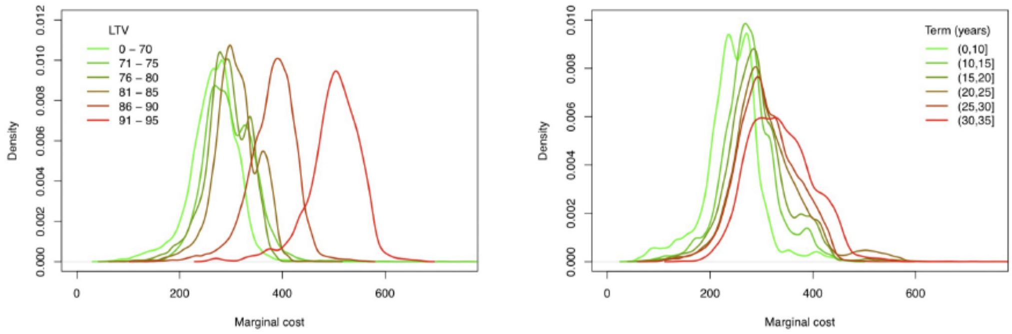
| # | | $z_b-$ | comb | | 1st quartile | 2nd quartile | 3rd quartile | % Median |
|---|---|---|---|---|---|---|---|---|
| | Age | Inc | FTB | Urb | | | | |
| 1 | L | L | Y | R | 100.53 (17.53) | 125.53 (26.37) | 150.53 (33.42) | 31.17% |
| 2 | H | L | Y | R | 28.93 (0.61) | 35.93 (0.60) | 42.93 (0.76) | 9.52% |
| 3 | L | H | Y | R | 33.11 (10.69) | 51.86 (15.22) | 70.62 (21.09) | 14.11% |
| 4 | H | H | Y | R | 24.36 (0.23) | 26.84 (0.38) | 29.32 (0.58) | 7.93% |
| 5 | L | L | N | R | 23.87 (0.56) | 29.50 (1.69) | 35.12 (2.96) | 8.76% |
| 6 | H | L | N | R | 27.93 (24.07) | 34.30 (34.72) | 40.67 (43.27) | 10.38% |
| 7 | L | H | N | R | 32.40 (23.03) | 52.29 (38.35) | 72.18 (47.98) | 16.67% |
| 8 | H | H | N | R | 30.36 (0.42) | 35.69 (0.68) | 38.19 (0.83) | 11.51% |
| 9 | L | L | Y | U | 20.98 (0.94) | 41.48 (1.98) | 45.22 (3.76) | 10.29% |
| 10 | H | L | Y | U | 25.16 (17.49) | 31.28 (21.29) | 37.41 (25.71) | 8.04% |
| 11 | L | H | Y | U | 25.46 (5.94) | 31.71 (21.09) | 37.96 (29.44) | 8.64% |
| 12 | H | H | Y | U | 19.61 (0.86) | 26.10 (5.05) | 31.35 (7.76) | 7.44% |
| 13 | L | L | N | U | 41.58 (14.65) | 61.50 (24.19) | 81.32 (30.87) | 18.06% |
| 14 | H | L | N | U | 32.08 (0.68) | 37.80 (0.83) | 40.53 (1.00) | 11.28% |
| 15 | L | H | N | U | 30.60 (0.43) | 33.03 (0.45) | 34.57 (0.51) | 10.56% |
| 16 | H | H | N | U | 20.72 (0.45) | 31.24 (0.28) | 34.01 (0.39) | 10.17% |

Table 2. Summary of quantiles search cost distributions.

Table 2 presents quartiles of nonparametrically estimated search cost distributions for 16 different borrower types (referred to as $z_b-$comb(inations). Age: L (below 30)/H (over 30). Inc(ome): L (below median)/H (above median). FTB (first time buyer status): Yes/No. Urb(an): U (urban area)/R (rural area). Columns 6-8 contains the median search cost in \pounds/month in the initial period. Column 9 expressed in relative terms (divided by median and average monthly payment, respectively). Bootstrap standard errors in parentheses based on 500 replications.

The results reveals several interesting findings: first, there are substantial differences between median search costs depending on different demographics, with young, low-income residents of rural areas who are purchasing their first home incurring as much as 31% of loan's monthly interest payment in search costs. Comparing relative magnitudes in the last column on the table also suggests, that in urban areas, first-time buyers spend relatively less on search than those that remortgage or buy a second property. This may suggest that search costs in this market might have a similar

Figure 5: Distributions of marginal costs



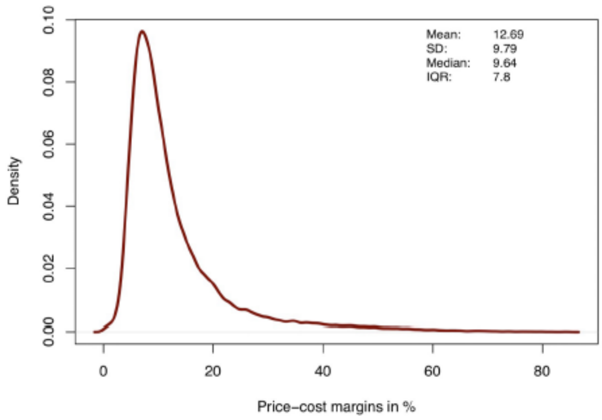

**Note:** Distributions of marginal costs for each conditioning variable in $z_l$ averaged over the remaining covariates. Left graph shows how higher loan-to-value ratio is associated with higher cost of providing a mortgage. Right graph shows a similar gradient in the duration of the loan.

interpretation to an opportunity cost of time and that learning does not seem to play a major role (i.e. subsequent experience in the mortgage market does not seem to reduce the costs of search efforts significantly). While we notice some discrepancies between the results in Table 2 and a corresponding table in MR, the relative magnitudes of search cost all fall in a similar range and the main qualitative conclusions remain unchanged, despite differences in sample composition and our use of parametric assumptions.

On the supply-side, we estimate distributions of marginal costs of providing loans of different types and evaluate the extent of banks' market power by analysing distributions of price-cost margins. While presenting a wide range of analyses is beyond the scope of this paper, we show selected results to prove that estimates derived from our model are in line with what is known about the mortgage market - e.g. that riskier loans (i.e. those with higher LTV ratio and longer crediting term) are costlier to supply. Figure Figure 5 nicely illustrate the latter. In particular, we simply used basic kernel density plots obtained by pooling $\left\{\widehat{R}_{im}^{(z_b,z_l)}\right\}_{i=1,m=1}^{I,M}$ to illustrate how they vary along a selected dimension of $z_l$. In this case, the graphs are a sanity check for what know about the mortgage market: riskier loans (those with higher LTV and longer crediting term) are costlier to supply. Note that those distributions are not the same as $h(\cdot|z_l)$ and are not a primitive of the model, since they implicitly integrate out the remaining dimensions of $z_l$.

We end this illustration with a plot of the distribution of percentage markups implied by the model estimates in Figure 6. Similarly to the results obtained using a fully nonparametric approach and a larger sample of lenders and loans in MR, the distribution is right-skewed with mean just above 10%. In our other work we explain how banks' market power is affected by the presence of intermediaries and look at changes in markups in a counterfactual without brokers. Equipped with the estimates of all model primitives, particularly the estimates of $H(\cdot|z_l)$ and $G(\cdot|z_b)$, the model can

Figure 6: Distributions of markups




**Note:** Distribution of markups defined as $100\% \times (p-c)/p$.

also be used to answer other policy-relevant questions, such as effects of bank mergers, introduction of LTV caps etc.[11]

# 8 Some Discussions

Our paper focuses on the theoretical and methodological aspects of identifying and estimating a search model using market shares and price data. The exposition of our theoretical models assumes each firm sells an identical product at the same price to all consumers. This posted-price framework is common in applications with retail data (e.g., online books (Hong and Shum (2006)), computer memory chips (Moraga-Gonzalez and Wildenbeest (2008)). Other potentially interesting applications in practice would involve individual specific price. Our mortgage application shows how aggregating prices enables the estimation procedure used for the posted-price framework to extend to the case where prices from the same firm vary across individuals. Particularly, we rely on the relation such as

$$\mathbb{E}\left[Y_{im}|P_{im}^{agg}\right] = \sum_{k=1}^{I} q_k \frac{k}{I}\left(1 - F\left(P_{im}^{agg}\right)\right)^{k-1}. \tag{29}$$

In the UK mortgage context, we argued using the median price as the as the representative market price is innocuous (Benetton (2021)). More generally, however, (29) can be thought of as a parametric assumption or approximation since it is not necessarily implied by the model unlike the posted-price case (cf. parts (ii) of Assumption D/DE1/DE2). Nevertheless, our parametric proposal is a

[11]The main counterfactual study in MR was to calculate the value of information provided by mortgage brokers, i.e. the change in consumer welfare based on new simulated equilibrium prices when intermediaries are removed from the market. This application is related to the counterfactual study in Salz, who analyzed value of the brokers in New York City's trade-waste market.

constructive approach in an empirical study because: one, the assumption has a clear interpretation in relating the market shares to a representative price; two, it is easy to use in a variety of context. It can be applied to the pure search model, with/out product differentiation, as well as a model with an intermediary; following the nonparametric or parametric estimation procedures explained in the paper.

We argue some assumptions are necessary for the feasible estimation of a search model with heterogeneous consumers and sellers from individualized price. Consider a special case, as studied by Salz (2020), when $H(\cdot)$ can be treated as known since he could identify $H(\cdot)$ from the auction data. Salz also showed nonparametric identification of the search cost distribution is possible. However, applying his simulated method of moments (SMM) estimator essentially requires a parametric specification on the cost distribution as well as moment selection. His method takes a full-solution approach that solves the model[12] for every parameter value during the optimization routine. A nonparametric version of his approach would involve flexible modelling of the cost distributions, e.g. via a sieve with a growing number of base functions, as well as letting the number of moments go to infinity asymptotically. We note that a different estimator that remains closer to the model than matching moments can also be constructed. Let $F^o(\cdot)$ denote the cdf of the observed price. As part of his identification proof, Salz showed that for all $p$,

$$1 - F^o(p) = \sum_{k=1}^{I} q_k \frac{k}{I} \left(1 - H\left(\beta^{-1}(p)\right)\right)^{k-1},$$

see equation (13)[13] in Appendix B.2.1 of his paper. He also showed $\beta^{-1}(\cdot)$ can be identified by solving a differential equation. The above equation resembles (29) and it indeed can be used to estimate $\mathbf{q}$ in closed form given $H\left(\beta^{-1}(\cdot)\right)$. The caveat is one needs to solve the differential equation (given on page 60 Salz (2020)). We expect this could be a formidable computational task in real applications, especially when price and cost distributions are parametric and depend on covariates, because the differential equation would have to be solved repeatedly during the optimization routine.

The point of the previous paragraph is to elucidate the challenge in estimating a search model like ours nonparametrically when prices are individualized even if it is theoretically possible to do so (noting this was made possible when $H(\cdot)$ is assumed to be known). This is perhaps not surprising because observing different prices offered by the same firm is suggestive that products it sells may not be homogeneous, which brings us into the realms of product differentiation. Specifically, as an

[12] The computational task can be more difficult when there are multiple equilibria. An advantage of our approach is we avoid this issue for the same reason as the so-called "two-step estimators", from the dynamic game literature, do. We refer the reader to the Introduction of Srisuma (2013) for a discussion.

[13] Salz uses $m$ instead of $k$. We note that the exponent term on the RHS of his expression should be $m-1$ and not $m$. This is just a minor typo with no consequence.

econometric model, it places us somewhere between a very simple posted-price framework with no supply side heterogeneity like Hong and Shum (2006) and, at the other extreme, a sophisticated model of differentiated product search in Moraga-González, Sándor, and Wildenbeest (2023) that requires a much richer data environment for identification than price and market shares alone (e.g., with individual search behaviour data). Therefore, the parametric approach we have shown, as well as Salz's, provide pragmatic ways to study search markets where products are not too heterogeneous with relatively modest requirements on data (i.e., prices and shares). Our model still paves the way to answer interesting counterfactual questions. E.g., Salz (2020) and Myśliwski and Rostom (2022) study the value of intermediaries in their respective applications. Other possible applications include quantification of effects on market price caused by changes to search costs (in the same vein as Choi, Dai, and Kim (2018), Moraga-González, Sándor, and Wildenbeest (2023)) or to production costs or market structure where the role of heterogenous firms can be emphasized.

We end by discussing some possible extensions to our work. One modelling extension we can consider is for firms to have different probabilities of being found by consumers. Our results readily extend to this case if we assume an equilibrium exists where the optimal pricing strategies of firms are strictly increasing and share the same support. We are unable to prove such equilibrium exists, but we are optimistic that it does based on positive results from the literature on asymmetric first-price auctions[14]. The challenge stems from the (quasi-)inverse of the optimal pricing strategies are solutions to a system of nonlinear differential equations that is difficult to analyze.[15]

On the econometrics, our nonparametric identification strategy readily extends to include observed heterogeneity. All the assumptions made in Sections 2-5 can be written to condition on covariates (that can include the number of firms). As in the auction model of Guerre et al. (2000), however, the nonparametric rate of convergence for the conditional distributions with continuous variables will be slower than that of the unconditional ones. In this case we expect the quantile regression approach of Gimenes and Guerre (2020), recently developed to mitigate the dimensionality issue in the auction literature, can be applicable to our search model. Finally, we do not deal with inference in this paper. Inference on the demand side parameters is relatively straightforward, e.g. see Sanches, Silva and Srisuma (2018). Establishing the asymptotic distribution and validity for the bootstrap of $\widehat{h}(\cdot)$ and $\widetilde{h}(\cdot)$ is more challenging. We conjecture this can be obtained by suitably adapting the arguments in a recent article by Ma, Marmer and Schneyerov (2019), where they derive

[14]Some existence results do exist, e.g. see Lebrun(1999) and Maskin and Riley (2000). Furthermore, a common support for the optimal bids in the first-price context is also known to hold (e.g. see Athey and Haile (2007).

[15]It is not trivial even to show existence of such equilibrium numerically. For instance, in a related problem, numerical studies of the equilibrium in asymmetric auctions is a current topic of research - e.g. see the discussion in Fibich and Gavish (2011).

the asymptotic variance for the GPV's estimator as well as showing inference using the bootstrap is valid.

# Appendix

This Appendix provides the proofs of lemmas and theorems. We omit the proofs of Lemmas 1, 2 and 10, Theorem 1, and the Propositions. These are either immediate consequences of what have discussed or proven in the main text. We also omit the proofs of Lemma 6 and Theorem 2 because they are very similar to the proofs of Lemma 9 and Theorem 3 respectively.

PROOF OF LEMMA 3. From (9), we have $\mathbb{E}[Y_{im}|X_{im}] = X_{im}^{\top}\mathbf{q}$. Multiply both sides by $X_{im}$ and take expectation yields $\mathbb{E}[X_{im}Y_{im}] = \mathbb{E}\left[X_{im}X_{im}^{\top}\right]\mathbf{q}$. Since $\mathbb{E}\left[X_{im}X_{im}^{\top}\right]$ has full rank, the proof follows from solving for $\mathbf{q}$. ■

PROOF OF LEMMA 4. By inspecting (4) and (5), $\beta(\cdot)$ is strictly increasing and continuously differentiable on $[\underline{R},\overline{R})$ therefore $\beta^{-1}(\cdot)\,(=\xi(\cdot))$ exists. Using the change-of-variable formula, we have $f(p) = \frac{h\left(\beta^{-1}(p)\right)}{\beta'\left(\beta^{-1}(p)\right)}$. From (5), we can write $f(p) = \psi\left(\beta^{-1}(p)\right)$, where $\psi(\cdot)$ is a real-value function defined on $[\underline{R},\overline{R})$ such that

$$\psi(r) = \frac{\left(\sum_{k=1}^{I} q_k k\,(1-H(r))^{k-1}\right)^2}{\left(\sum_{k=2}^{I} q_k k\,(k-1)\,(1-H(r))^{k-2}\right)\left(\sum_{k=1}^{I} q_k k \int_{s=r}^{\overline{R}} (1-H(s))^{k-1}\,ds\right)}. \tag{30}$$

Part (a) follows from $\inf_{p\in[\underline{P},\overline{P}]} f(p) = \inf_{r\in[\underline{R},\overline{R}]} \psi(r) \geq \frac{q_1^2}{\left(\sum_{k=2}^{I} q_k k(k-1)\right)\left(\overline{R}\sum_{k=1}^{I} q_k k\right)} > 0$. Part (b) follows from $\lim_{p\to\overline{P}} f(p) = \lim_{r\to\overline{R}} \psi(r) = \infty$. To obtain the expression in part (c), we know that $\beta(r)$ is the maximizer of the following function,

$$\Lambda(p,r) = (p-r)\sum_{k=1}^{I} q_k \frac{k}{I}\left(1-H(\xi(p))\right)^{k-1},$$

for any $r$. $\beta(r)$ is also the zero to $\frac{\partial}{\partial p}\Lambda(p,r)$, where

$$\begin{aligned}\frac{\partial}{\partial p}\Lambda(p,r) &= \sum_{k=1}^{I} q_k \frac{k}{I}\left(1-H(\xi(p))\right)^{k-1} \\ &\quad -(p-r)\,\xi'(p)\,h(\xi(p))\sum_{k=2}^{I} q_k \frac{k(k-1)}{I}\left(1-H(\xi(p))\right)^{k-2}.\end{aligned}$$

Note that the cdf and pdf of $P_{im}$ and $R_{im}$ are related through $F(p) = H(\xi(p))$ and $f(p) = \xi'(p) h(\xi(p))$ respectively. Substitute these in and impose the first-order condition leads to,

$$\sum_{k=1}^{I} q_k k (1 - F(p))^{k-1} = (p - \xi(p)) f(p) \sum_{k=2}^{I} q_k k (k-1) (1 - F(p))^{k-2}.$$

Rearranging the relation above gives (10).■

Proof of Lemma 5. From (5), we see that $\beta^{-1}(\cdot)$ is $\tau + 1$ times continuously differentiable on $[\underline{P}, \overline{P})$ as $\beta'(\cdot) > 0$ on $[\underline{R}, \overline{R})$. The result then follows from the fact that $\psi(r)$, see (30), is a smooth functional of $H(\cdot)$ for all $r \in [\underline{R}, \overline{R})$. ■

Proof of Lemma 7. For any $p \in [\underline{P}, \overline{P}]$ there exists a unique $r \in [\underline{R}, \overline{R}]$ such that for some $\widetilde{r} \in (\underline{R}, \overline{R})$,

$$\left(\overline{P} - p\right) f(p) = \beta'(\widetilde{r}) \left(\overline{R} - r\right) f(\beta(r)) = \left(\overline{R} - r\right) h(\widetilde{r}) f(\beta(r)).$$

The first equality comes from replacing $\left(p, \overline{P}\right)$ with $\left(\beta(r), \beta\left(\overline{R}\right)\right)$ and applying the Mean Value Theorem. The second equality uses the change-of-variable relation between $f(\cdot)$ and $h(\cdot)$. The result follows as we let $r \to \overline{R}$ due to the regularity of $h(\cdot)$ at $\overline{R}$ (i.e., existence of $\lim_{r \to \overline{R}} \left(\overline{R} - r\right) h(r)$) and the continuity of $\beta(\cdot)$.■

Proof of Lemma 8. For part (a), given that $f^{\dagger}\left(p^{\dagger}\right) = \exp\left(-p^{\dagger}\right) f\left(\overline{P} - \exp\left(-p^{\dagger}\right)\right)$ for all $p^{\dagger} \in [-\ln\left(\overline{P} - \underline{P}\right), \infty)$, it suffices to show $\lim_{p \to 0} p f\left(\overline{P} - p\right) = 0$. This is in fact an implication of the regularity of $f(\cdot)$ at $\overline{P}$. The same idea of proof used in Proposition 2(i) of Srisuma (2023) applies. Part (b), the fact that $f^{\dagger}(\cdot)$ has the same degree of smoothness as $f(\cdot)$ can be proven as in Proposition 5 of Srisuma (2023). The boundedness of the derivatives can be proven as in Proposition 2(ii) of Srisuma (2023).■

Proof of Lemma 9. From (18) when $\widetilde{R}_{im} < \infty$ we can write,

$$\begin{aligned}
\widetilde{R}_{im} - R_{im} &= I_1(P_{im}) + I_2(P_{im}), \text{ where} \\
I_1(P_{im}) &= \Psi\left(\widehat{\mathbf{q}}, \widetilde{f}(P_{im}), \widehat{F}(P_{im})\right) - \Psi\left(\mathbf{q}, \widetilde{f}(P_{im}), F(P_{im})\right), \\
I_2(P_{im}) &= \Psi\left(\mathbf{q}, \widetilde{f}(P_{im}), F(P_{im})\right) - \Psi\left(\mathbf{q}, f(P_{im}), F(P_{im})\right),
\end{aligned}$$

where $\Psi(\mathbf{q}, f(P_{im}), F(P_{im})) = \frac{\sum_{k=1}^{I} q_k k (1 - F(P_{im}))^{k-1}}{f(P_{im}) \sum_{k=2}^{I} q_k k (k-1)(1 - F(P_{im}))^{k-2}}$ so that $\Psi\left(\widehat{\mathbf{q}}, \widetilde{f}(P_{im}), \widehat{F}(P_{im})\right)$ and $\Psi\left(\mathbf{q}, \widetilde{f}(P_{im}), F(P_{im})\right)$ are estimated counterparts of $\Psi(\mathbf{q}, f(P_{im}), F(P_{im}))$ where some or all components of $(\mathbf{q}, f(P_{im}), F(P_{im}))$ are replaced by $\left(\widehat{\mathbf{q}}, \widetilde{f}(P_{im}), \widehat{F}(P_{im})\right)$ accordingly. By Lemma 4(a), we know $\inf_{p \in \left[\underline{P} + \delta_M, \overline{P} - \delta_M\right]} \widetilde{f}(p) > c_0$ for some $c_0 > 0$ with probability approaching one as $M \to \infty$.

Given the convergence rates in Propositions 3 and 4, it is straightforward to verify that the partial derivatives of $\Psi\left(\mathbf{q}, \widetilde{f}(P_{im}), F(P_{im})\right)$ with respect to its first and third arguments are also almost surely uniformly bounded. Therefore by the mean value theorem it follows that

$$|I_1(P_{im})| = O_p\left(\|\widehat{\mathbf{q}} - \mathbf{q}\| + \sup_{p\in\left[\underline{P}+\delta_M, \overline{P}-\delta_M\right]}\left|\widehat{F}(p) - F(p)\right|\right)$$

So that $|I_1(P_{im})| = o(\eta_M)$ almost surely. For $I_2$, we can write

$$I_2(P_{im}) = -\left(\frac{\widetilde{f}(P_{im}) - f(P_{im})}{\widetilde{f}(P_{im}) f(P_{im})}\right)\frac{\sum_{k=1}^{I} q_k k\left(1 - F(P_{im})\right)^{k-1}}{\sum_{k=2}^{I} q_k k(k-1)\left(1 - F(P_{im})\right)^{k-2}},$$

so that

$$|I_2(P_{im})| = O\left(\sup_{p\in\left[\underline{P}+\delta_M, \overline{P}-\delta_M\right]}\left|\widetilde{f}(p) - f(p)\right|\right)\ a.s.$$

The upper bounds for $|I_1(P_{im})|$ and $|I_2(P_{im})|$ are independent of $P_{im}$. The proof then follows from applying the convergence rates of the quantities in $|I_1(P_{im})|$ and $|I_2(P_{im})|$ as stated in Proposition 4.■

Proof of Theorem 3. From (19),

$$\begin{aligned}
\widetilde{h}(r) - h(r) &= J_1(r) + J_2(r) + J_3(r),\ \text{where} \\
J_1(r) &= \frac{1}{MIb_{h,M}}\sum_{m=1}^{M}\sum_{i=1}^{I}\left(K\left(\frac{\widetilde{R}_{im} - r}{b_{h,M}}\right) - K\left(\frac{R_{im} - r}{b_{h,M}}\right)\right)\mathbf{1}\left[\widetilde{R}_{im} < \infty\right], \\
J_2(r) &= -\frac{1}{MIb_{h,M}}\sum_{m=1}^{M}\sum_{i=1}^{I}K\left(\frac{R_{im} - r}{b_{h,M}}\right)\mathbf{1}\left[\widetilde{R}_{im} = \infty\right], \\
J_3(r) &= \frac{1}{MIb_{h,M}}\sum_{m=1}^{M}\sum_{i=1}^{I}K\left(\frac{R_{im} - r}{b_{h,M}}\right) - h(r).
\end{aligned}$$

For $J_1$:

$$J_1(r) = \frac{1}{MIb_{h,M}}\sum_{m=1}^{M}\sum_{i=1}^{I}\left(K'\left(\frac{R_{im} - r}{b_{h,M}}\right)\left(\frac{\widetilde{R}_{im} - R_{im}}{b_{h,M}}\right) + \frac{1}{2}K''\left(\frac{\overline{R}_{im} - r}{b_{h,M}}\right)\left(\frac{\widetilde{R}_{im} - R_{im}}{b_{h,M}}\right)^2\right)\mathbf{1}\left[\widetilde{R}_{im} < \infty\right]$$

where $\overline{R}_{im}$ is some mid-point between $\widetilde{R}_{im}$ and $R_{im}$. Then we have

$$\begin{aligned}
|J_1(r)| \leq\ & \frac{\sup_{i,m\ \text{s.t.}\ \widetilde{R}_{im}<\infty}\left|\widetilde{R}_{im} - R_{im}\right|}{b_{h,M}}\frac{1}{MIb_{h,M}}\sum_{m=1}^{M}\sum_{i=1}^{I}\left|K'\left(\frac{R_{im} - r}{b_{h,M}}\right)\right| \\
& + \frac{\left(\sup_{i,m\ \text{s.t.}\ \widetilde{R}_{im}<\infty}\left|\widetilde{R}_{im} - R_{im}\right|\right)^2}{b_M^3}\frac{1}{2MI}\sum_{m=1}^{M}\sum_{i=1}^{I}\mathbf{1}\left[\widetilde{R}_{im} < \infty\right]\sup_{v\in\mathbb{R}}K''(v).
\end{aligned}$$

It can be shown using standard methods for kernel estimators that

$$\sup_{r\in\left[\underline{R}+\varsigma_M,\overline{R}-\varsigma_M\right]}\left|\frac{1}{MIb_{h,M}}\sum_{m=1}^{M}\sum_{i=1}^{I}\left|K'\left(\frac{R_{im}-r}{b_{h,M}}\right)\right|-h\left(r\right)\int\left|K'\left(v\right)\right|dv\right|=o\left(1\right),$$

where $\sup_{r\in\left[\underline{R},\overline{R}\right]}h\left(r\right)\int\left|K'\left(v\right)\right|dv$ is finite. Since $\left[\widetilde{R}_{im}<\infty\right]$ is an almost sure set asymptotically, $\frac{1}{MI}\sum\limits_{m=1}^{M}\sum\limits_{i=1}^{I}\mathbf{1}\left[\widetilde{R}_{im}<\infty\right]$ converges to 1 almost surely and,

$$\frac{1}{2MI}\sum_{m=1}^{M}\sum_{i=1}^{I}\mathbf{1}\left[\widetilde{R}_{im}<\infty\right]\sup_{v\in\mathbb{R}}K''\left(v\right)=\frac{1}{2}\sup_{v\in\mathbb{R}}K''\left(v\right)+o\left(1\right).$$

It follows that

$$\sup_{r\in\left[\underline{R}+\varsigma_M,\overline{R}-\varsigma_M\right]}\left|J_1\left(r\right)\right|\leq O\left(\frac{\eta_M}{b_{h,M}}+\frac{\eta_M^2}{b_M^3}\right).$$

When $\eta_M=O\left(b_M^2\right)$ it follows that,

$$\sup_{r\in\left[\underline{R}+\varsigma_M,\overline{R}-\varsigma_M\right]}\left|J_1\left(r\right)\right|\leq O\left(\frac{\eta_M}{b_{h,M}}\right)\ a.s.$$

For $J_2$, since $\left[\widetilde{R}_{im}=\infty\right]$ is a null set asymptotically and $\mathbf{1}\left[\widetilde{R}_{im}=\infty\right]=o\left(\upsilon_M\right)$, by choosing $\upsilon_M=o\left(\frac{\eta_M}{b_{h,M}}\right)$,

$$\sup_{r\in\left[\underline{R}+\varsigma_M,\overline{R}-\varsigma_M\right]}\left|J_2\left(r\right)\right|\leq o\left(\upsilon_M\right)\ a.s.$$

For $J_3$, it is a standard result in kernel estimation that

$$\sup_{r\in\left[\underline{R}+\varsigma_M,\overline{R}-\varsigma_M\right]}\left|J_3\left(r\right)\right|=O\left(b_{h,M}^{\tau}+\eta_M^{*}\right)\ a.s.$$

The bias component in $J_3$ is of the same order as $\frac{\eta_M^*}{b_{h,M}}=o\left(\frac{\eta_M}{b_{h,M}}\right)$ and the stochastic part is also $o\left(\frac{\eta_M}{b_{h,M}}\right)$. ■

PROOF OF THEOREM 4. It suffices to provide the best responses of consumers and firms analogous to those in Lemmas 1 and 2 respectively, and give the definition of a quasi-symmetric equilibrium. These results are stated in Lemmas 12 and 13 below. In particular, Lemma 11 replaces $\{\Delta_k\}_{k=1}^{I-1}$ in equation (3) by $\{\Upsilon_k\}_{k=1}^{I-1}$ and Lemma 12 use the distribution of random utilities based on (22) instead of prices.

LEMMA 12. *Suppose Assumption D' holds. Then the consumer's best response is a map* $\sigma_D:\mathcal{L}\rightarrow\mathbb{S}^{I-1}$ *such that for any* $L$ *in* $\mathcal{L}$,

$$\sigma_D\left(L\right)=\left\{\begin{array}{cc}1-G\left(\Upsilon_k\left(L\right)\right) & \text{for } k=1\\ G\left(\Upsilon_{k-1}\left(L\right)\right)-G\left(\Upsilon_k\left(L\right)\right) & \text{for } k>1\end{array}\right.. \quad (31)$$

Lemma 13. *Suppose Assumption D' holds. Then the firm's best response is a map* $\sigma_S : \mathbb{S}^{I-1} \to \mathcal{L}$ *such that for any* $\mathbf{q}$ *in* $\mathbb{S}^{I-1}$, $\sigma_S(\mathbf{q})$ is the cdf of $\mu(x(R_{0i});\mathbf{q})$ *where* $\mu(x(\cdot);\mathbf{q})$ *is defined as in (22).*

We can now define a quasi-symmetric equilibrium as follows.

Definition 2 . *A pair* $(\mathbf{q}, L) \in \mathbb{S}^{I-1} \times \mathcal{L}$ *is a quasi-symmetric equilibrium if* $\mathbf{q} = \sigma_D(L)$ *and* $L = \sigma_S(\mathbf{q})$, *where* $\sigma_S(\cdot)$ *and* $\sigma_S(\cdot)$ *are defined in Lemmas 10 and 11 respectively.*

The proof of the theorem follows immediately from here. ■

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